On Improving Uspensky-Sherman's Normal Approximation by an Edgeworth-Expansion Approximation

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Abstract

The exact bound of the remainder in normal approximation obtained by Uspensky (1937) was sharpened by Sherman (1971) for the sample mean from a continuous uniform distribution. Their exact bounds of $O(n^{-1})$ is now improved to an exact bound of $O(n^{-2})$ on the remainder after one-step higher-order Edgeworth-expansion approximation. The estimations of the error obtained from the improved bound is so sharp that it may provide practically useful information in statistical applications.

1 Introduction.

Let $X_1, X_2, \cdots, X_n$ be i.i.d. (independently and identically distributed) rv's (random variables) having mean zero and finite variance, i.e., $EX^2 \equiv \sigma^2 < \infty$. We consider the normalized sample mean $T_n = (\sigma/\sqrt{n})^{-1}\bar{X}_n$, where $\bar{X}_n = n^{-1}\sum_{j=1}^{n} X_j$. Denote its cdf (cumulative distribution function) and the standard normal cdf, by $F_n(x) = P(T_n \le x)$ and $\Phi(x)$, respectively; and put $\Delta_n = \sup_{x\in\Re} |F_n(x) - \Phi(x)|$. Then, $\lim_{n\to\infty} \Delta_n = 0$ by the well-known CLT. This normal approximation is frequently used in statistical applications and it has been justified typically by the reference to the CLT (Central Limit Theorem). Unfortunately, this simple asymptotic

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normality does not provide any information on how good the normal approximations are. In this regard, the problem of estimating the remainder term is essential in applications of the CLT. The celebrated Berry (1941)-Esséen (1942) inequality ensures, under the existence of the third moment, that for all $n \geq 1$ and for all x

$$|F_n(x) - \Phi(x)| \leq \delta_{1,n} n^{-1/2} \tag{1.1}$$

where $\delta_{1,n} = CE|X|^3(EX^2)^{-3/2}$ and C is a universal absolute constant, called the Berry-Esséen constant, of which the best estimate is 0.7655 (for i.i.d. case) obtained by Shiganov (1982), improving over the well-known van Beek's (1972) constant 0.7975. Unfortunately, the Berry-Esseen bound $\delta_{1,n}$ is so crude that, in general, it does not help us much in daily statistical practices (such as estimating necessary sample sizes assuring desired accuracy in normal approximations).

On the other hand, for the special case of the uniform distribution, Uspensky (1937, p.305) obtained very sharp estimation of the error, i.e., for all $n \geq 1$ and for all x,

$$|F_n(x) - \Phi(x)| \leq n^{-1}\left(\frac{2}{15} + \frac{n}{\pi}\left(\frac{2}{\pi}\right)^n + \frac{12}{\pi^3}e^{-\pi^2 n/24}\right), \tag{1.2}$$

which (the constant 2/15 should read $2/(15\pi)$, considered as an apparent mistake by Uspensky) was improved by Sherman (1971). In fact, for all $n \geq 1$ and for all x,

$$|F_n(x) - \Phi(x)| \leq \delta_{2,n} n^{-1} \tag{1.3}$$

where

$$\delta_{2,n} \equiv \left(\frac{2}{15\pi} - \left(\frac{\pi n}{180} + \frac{2}{15\pi}\right)e^{-\pi^2 n/24} + \frac{1}{\pi}\left(\frac{2}{\pi}\right)^n + \frac{12}{\pi^3}e^{-\pi^2 n/24}\right). \tag{1.4}$$

(We note that Sherman's equation (2) is in error, the constant of 1/2 should be added to its right hand side).

Under the symmetry, this estimation of $O(n^{-1})$ over Berry-Esséen's of $O(n^{-1/2})$ is impressive and, in fact, due to the existence of Edgeworth expansion. The higher-order Edgeworth expansions under suitable assumptions were justified rigorously by the classical theory on sums of independent rv's. The interested readers are referred to the monographs of Cramér (1970), Gnedenko and Kolmogorov (1968), Feller (1971), Petrov (1975, 1995), Bhattacharya and Rao (1976), Bhattacharya and Denker (1990), Hall (1992), and Ghosh (1994); among many other important works cited therein.

Since remainders are expressed in $O(\cdot)$ and/or $o(\cdot)$-terms in the general theory, deriving any finite-sample estimation on them is quite challenging in general. For the special case that Uspensky and Sherman dealt with, we now, in Section 2, improve their bound of (1.3) by one-step higher-order Edgeworth expansion with an exact bound on the remainder. In Section 3, we provide some comments on numerical comparisons of our improved approximation with those of Berry-Esséen and Uspensky-Sherman.

2 An Exact Bound on the Fourth-Order Edgeworth-Expansion Approximation.

Let $X_1, X_2, \cdots, X_n$ be i.i.d. rv's (random variables) having a continuous uniform distribution. Without loss of generality, we may assume that the common density is given by $f(x) = (2A)^{-1}$ or 0, according as $|x| \le A$ or $|x| > A$. Hence $\sigma^2 = A^2/3$. We now consider the normalized sample mean

$$T_n = \sqrt{n}\sigma^{-1}\bar{X}_n = n^{-1/2}\sum\nolimits_{j=1}^{n} Z_j \tag{2.1}$$

where Z_j's are uniformly distributed over $(-\sqrt{3}, \sqrt{3})$. The third cumulant of T_n is zero, because of symmetry; while, its fourth cumulant, multiplied by n, is $\kappa_4 = EX^4 - 3 = -6/5$. We note that the ch.f. (characteristic function) of T_n is

$$\varphi_n(t) = Ee^{itT_n} = \prod\nolimits_{j=1}^{n} Ee^{it(\Delta n^{1/2})^{-1}\sqrt{3}Y_j} = \left(\frac{\sin(\sqrt{3}n^{-1/2}t)}{\sqrt{3}n^{-1/2}t}\right)^n. \tag{2.2}$$

Denoting $F_n(x) = P(T_n \le x)$, we consider its formal Edgeworth expansion of order 4, i.e.,

$$\hat{F}_{4,n}(x) = \Phi(x) - \phi(x)(x^3 - 3x)\kappa_4(24n)^{-1}, \tag{2.3}$$

of which the Fourier transform is given by

$$\hat{\varphi}_{4,n}(t) = e^{-t^2/2}(1 + \kappa_4(24n)^{-1}t^4). \tag{2.4}$$

Put $R(t) = F_n(x) - \hat{F}_{4,n}(x)$ and let $r(t)$ be its Fourier transform, then

$$\int_{-\infty}^{\infty} \left|t^{-1}r(t)\right|dt = \int_{-\infty}^{\infty} \left|t^{-1}(\varphi_n(t) - \hat{\varphi}_{4,n}(t))\right|dt < \infty \tag{2.5}$$

as is estimated in the rest of this section. Thus, applying Theorem I.3.7 in Petrov (1975, p. 13), we can express and estimate the remainder in approximating $F_n(x)$ by the Edgeworth expansion $\hat{F}_{4,n}(x)$ as:

$$\left|F_n(x) - \hat{F}_{4,n}(x)\right| \le \frac{1}{\pi}\int_0^{\infty} t^{-1}\left|\varphi_n(t) - \hat{\varphi}_{4,n}(t)\right| dt \le I_1 + I_2 + I_3 \tag{2.6}$$

where

$$\begin{aligned} I_1 &= \pi^{-1}\int_0^{(\pi/2)n^{1/2}/\sqrt{3}} t^{-1}\left|\varphi_n(t) - \hat{\varphi}_{4,n}(t)\right| dt, \\ I_2 &= \pi^{-1}\int_{(\pi/2)n^{1/2}/\sqrt{3}}^{\infty} t^{-1}\left|\varphi_n(t)\right| dt, \\ I_3 &= \pi^{-1}\int_{(\pi/2)n^{1/2}/\sqrt{3}}^{\infty} t^{-1}\left|\hat{\varphi}_{4,n}(t)\right| dt. \end{aligned} \tag{2.7}$$

We now need estimates of these three integrals. To this end, we first note that

$$e^{-x^2/6 - x^4/180 - x^6/2295} \le \sin x/x \le e^{-x^2/6}(1 - x^4/180) \le e^{-x^2/6 - x^4/180}$$

for $|x| \le \pi/2$. Then, taking $x = \sqrt{3}n^{-1/2}t$ and using (2.2), we obtain that, for $|t| \le (\pi/2)n^{1/2}/\sqrt{3}$,

$$e^{-t^2/2 - (20n)^{-1}t^4 - (85n^2)^{-1}t^6} \le \varphi_n(t) \le e^{-t^2/2 - (20n)^{-1}t^4}. \tag{2.8}$$

Applying inequalities: $1-e^{-x} \le x$ for all real x and $e^{-x}-1+x \le (1/2)x^2$ for $x \ge 0$; and noting that

$$e^{-t^2/2}(1-t^4/(20n)) = \hat{\varphi}_{4,n}(t) \le e^{-t^2/2-t^4/(20n)}$$

we obtain

$$\begin{aligned}\varphi_n(t)-\hat{\varphi}_{4,n}(t) &\ge -e^{-t^2/2-t^4/(20n)}(1-e^{-t^6/(85n^2)}) \ge -e^{-t^2/2}(1-e^{-t^6/(85n^2)})\\ &\ge -e^{-t^2/2}(t^6/(85n^2)) \ge -e^{-t^2/2}(t^6/(85n^2) \vee 2^{-1}(t^4/(20n))^2)\end{aligned}$$

and

$$\begin{aligned}\varphi_n(t)-\hat{\varphi}_{4,n}(t) &\le e^{-t^2/2}(e^{-t^4/(20n)}-1+t^4/(20n))\\ &\le e^{-t^2/2}2^{-1}(t^4/(20n))^2 \le e^{-t^2/2}(t^6/(85n^2) \vee 2^{-1}(t^4/(20n))^2).\end{aligned}$$

These two inequalities ensure that, for $|t| \le (\pi/2)n^{1/2}/\sqrt{3}$,

$$\begin{aligned}|\varphi_n(t)-\hat{\varphi}_{4,n}(t)| &\le e^{-t^2/2}\left(\frac{t^6}{85n^2} \vee \frac{1}{2}\left(\frac{t^4}{20n}\right)^2\right)\\ &= n^{-2}e^{-t^2/2}\left(\frac{1}{85}t^6 \vee \frac{1}{800}t^8\right). \qquad (2.9)\end{aligned}$$

Putting $\alpha = (\pi/2)n^{1/2}/\sqrt{3}$ and denoting $J_k(\alpha) = \int_0^\alpha t^k e^{-t^2/2}dt$, $k = 1, 2, \cdots$, we obtain

$$\begin{aligned}I_1 &= \pi^{-1}\int_0^\alpha t^{-1}\left|\varphi_n(t)-\hat{\varphi}_{4,n}(t)\right| dt\\ &\le n^{-2}\pi^{-1}\int_0^\alpha e^{-t^2/2}\left(\frac{1}{85}t^5 \vee \frac{1}{800}t^7\right) dt\\ &\le n^{-2}\frac{1}{85\pi}J_5(\alpha) \equiv I_1^* n^{-2} \qquad (2.10)\end{aligned}$$

where

$$I_1^* \equiv \frac{1}{85\pi}J_5(\alpha) = \frac{8}{85\pi} - \frac{1}{\pi}\left(\frac{8}{85}+\frac{\pi^2 n}{85\cdot 3}+\frac{\pi^4 n^2}{85\cdot 144}\right)e^{-\pi^2 n/24}.$$

We now estimate the remaining two integrals I_2 and I_3 as:

$$I_2 = \pi^{-1}\int_\alpha^\infty t^{-1}\left|\varphi_n(t)\right| dt \le \pi^{-1}\int_{\pi/2}^\infty y^{-n-1}dt = \frac{1}{\pi n}\left(\frac{2}{\pi}\right)^n \equiv I_2^* n^{-2} \qquad (2.11)$$

and

$$\begin{aligned}I_3 &= \pi^{-1}\int_\alpha^\infty t^{-1}\left|\hat{\varphi}_{4,n}(t)\right| dt \le \pi^{-1}\left(\int_\alpha^\infty t^{-1}e^{-t^2/2}dt + \frac{|\kappa_4|}{24n}\int_\alpha^\infty t^3 e^{-t^2/2}dt\right)\\ &= \pi^{-1}\left(\int_\alpha^\infty t^{-1}e^{-t^2/2}dt + \frac{|\kappa_4|}{24n}(2+\alpha^2)e^{-\alpha^2/2}\right)\\ &\le \pi^{-1}\left(\alpha^{-2}e^{-\alpha t^2/2} + \frac{|\kappa_4|}{24n}(2+\alpha^2)e^{-\alpha^2/2}\right) \equiv I_3^* n^{-2} \qquad (2.12)\end{aligned}$$

where

$$I_2^* \equiv \frac{n}{\pi}\left(\frac{2}{\pi}\right)^n,$$

$$\begin{aligned} I_3^* &\equiv \frac{n^2}{\pi}\alpha^{-2}e^{-\alpha^2/2} + \frac{|\kappa_4|n}{24\pi}(2+\alpha^2)e^{-\alpha^2/2} \\ &= \frac{1}{\pi}\left(\frac{12n}{\pi^2} + \frac{|\kappa_4|\,n}{12} + \frac{|\kappa_4|\,\pi^2n^2}{288}\right)e^{-\pi^2 n/24}. \end{aligned}$$

Finally, it follows from (2.6), (2.10), (2.11) and (2.12) that, for all $n \geq 1$ and all x,

$$\left|F_n(x) - \hat{F}_{4,n}(x)\right| \leq (I_1^* + I_2^* + I_3^*)n^{-2} \equiv \delta_{4,n}n^{-2} \tag{2.13}$$

where

$$\begin{aligned} \delta_{4,n} = \; & \frac{8}{85\pi} - \frac{1}{\pi}\left(\frac{8}{85} + \frac{\pi^2 n}{85\cdot 3} + \frac{\pi^4 n^2}{85\cdot 144}\right)e^{-\pi^2 n/24} \\ & + \frac{n}{\pi}\left(\frac{2}{\pi}\right)^n + \frac{1}{\pi}\left(\frac{12n}{\pi^2} + \frac{|\kappa_4|\,n}{12} + \frac{|\kappa_4|\,\pi^2 n^2}{288}\right)e^{-\pi^2 n/24}. \end{aligned} \tag{2.14}$$

3 Numerical Computations and Discussions.

The Uspensky-Sherman bound of (1.3) and our bound of (2.13) provide precise estimates of the remainders of $O(n^{-1})$ and $O(n^{-2})$, respectively, for each sample size n. A comparison of these bounds, including the Berry-Esséen bound, is provided by Figures at the end of this note.

For the purpose of a concise comparison, we consider two types of constants. In the spirit of Berry-Esséen, we first consider constants A_r^* satisfying the inequality, for all $n \geq 1$,

$$\sup_{x\in\Re}\left|F_n(x) - \hat{F}_{r,n}(x)\right| \leq A_r^* n^{-r/2} \tag{3.1}$$

and, second, constants A_{r,n_*} (with a given sample size n_*) satisfying that, for all $n \geq n_*$,

$$\sup_{x\in\Re}\left|F_n(x) - \hat{F}_{r,n}(x)\right| \leq A_{r,n_*} n^{-r/2} \tag{3.2}$$

where $\hat{F}_{r,n}(x) = \Phi(x)$ for $r = 1,2$; while $\hat{F}_{r,n}(x) = \Phi(x) - \phi(x)(x^3 - 3x)\kappa_4(24n)^{-1}$ for $r = 3,4$ (i.e., for our refined approximation). Note that $A_1^* = 0.7655(3\sqrt{3}/4) \approx 0.9945$ is due to Shiganov (1982) and that the constant A_r^* appearing in (3.1) is usually hidden behind $O(n^{-r/2})$-term expression.

It follows from Uspensky-Sherman bound that, for all $n \geq 1$ and real x,

$$|F_n(x) - \Phi(x)| \leq \delta_{2,n}n^{-1} \tag{3.3}$$

with $\delta_{2,n}$ given by (1.4). This bound yields that, for all $n \geq 1$ and real x,

$$|F_n(x) - \Phi(x)| \leq A_2^* n^{-1} \tag{3.4}$$

where the absolute constant A_2^* is given by

$$A_2^* \equiv \frac{1}{\pi}\left(\frac{2}{15} - \left(\frac{\pi^2}{180} + \frac{2}{15}\right)e^{-\pi^2/24} + \frac{2}{\pi} + \frac{12}{\pi^2}e^{-\pi^2/24}\right) \approx \pi^{-1}(1.45114) \approx 0.46192.$$

However, by direct numerical computations using (1.3), we obtain that, for all $n \geq n_*$,

$$|F_n(x) - \Phi(x)| \leq A_{2,n_*}n^{-1} \tag{3.5}$$

where, for examples, $A_{2,1} = 0.46192$, $A_{2,10} = 0.04871$, $A_{2,20} = 0.04248$, $A_{2,30} = 0.04245$.

On the other hand, it follows by our extended bound that, for all $n \geq 1$ and real x,

$$\left|F_n(x) - \Phi(x) + \phi(x)(x^3 - 3x)\kappa_4(24n)^{-1}\right| \leq n^{-2}\delta_{4,n} \tag{3.6}$$

with $\delta_{4,n}$ given by (2.14). This yields an inequality corresponding to (3.4). More precisely, since numerical computations show that the maximum of the sum $I_1^* + I_2^* + I_3^*$ of (2.13) occurs when $n = 3$, we obtain that, for all $n \geq 1$ and real x,

$$\left|F_n(x) - \Phi(x) + \phi(x)(x^3 - 3x)\kappa_4(24n)^{-1}\right| \leq A_4^* n^{-2} \tag{3.7}$$

with

$$A_4^* \equiv \pi^{-1}(1.65713) \approx 0.52749$$

Furthermore, by direct numerical computations using the right hand side of (2.13), i.e., $\delta_{4,n}n^{-2}$, we obtain that, for all $n \geq n_*$,

$$\left|F_n(x) - \Phi(x) + \phi(x)(x^3 - 3x)\kappa_4(24n)^{-1}\right| \leq A_{4,n_*} n^{-2} \tag{3.8}$$

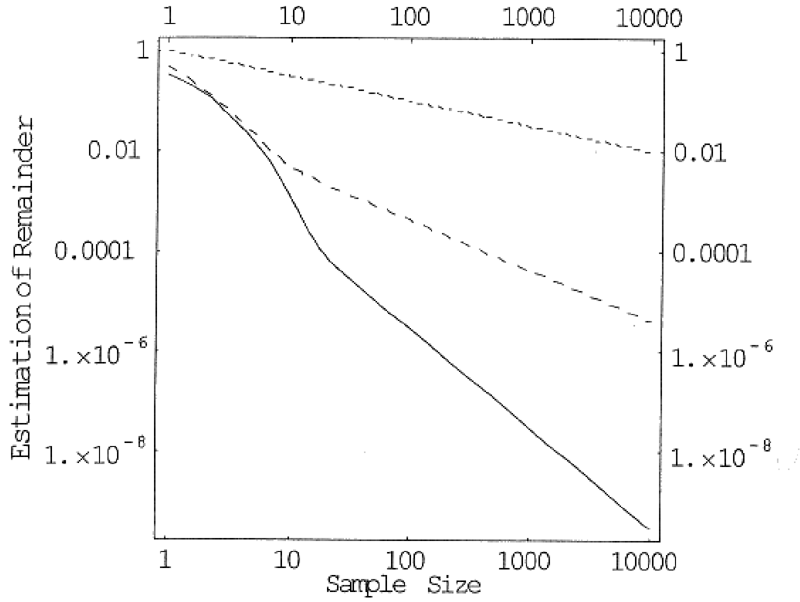
where, for examples, $A_{4,1} = 0.52749$, $A_{4,10} = 0.13726$, $A_{4,20} = 0.03382$ and $A_{4,30} = 0.03007$.

On viewing $A_2^* = 0.4620$ and $A_4^* = 0.5275$ as well as comparing A_{2,n_*} with A_{4,n_*}, we note an interesting fact. Our estimates of the constants following our best available means for approximating the remainders, the constants associated with higher-order approximation is larger in the beginning as the order of Edgeworth expansions is higher. This fact is clearly shown in the graphs in Figure 3.1; i.e., our remainder does not perform $O(n^{-2})$ rate of convergence until the sample size becomes at least larger than $n = 10$.

A general method of estimating integrals, corresponding to I_2, for many other important statistics is not available. If no significantly better approximation could be provided than that of (2.11), then its contribution to the constant A_r^* would be $\sup_{1\leq n<\infty} n^{r/2-1}(2^n\pi^{-(n+1)})$. This supremum turns out to be: 0.2027, 0.2581, 0.8633, 4.628, 35.88, 356.9, 4335, 62198, $(1.04)10^6$, $(1.95)10^7$, for $r = 2, 4, 6, 8, \cdots, 20$, respectively. Quite contrary to our innocent perception obtained from $O(n^{-r/2})$-expression of remainders, higher-order asymptotic approximations would estimate the target quantity truly up to the claimed degree of accuracy only when the sample size is also larger.

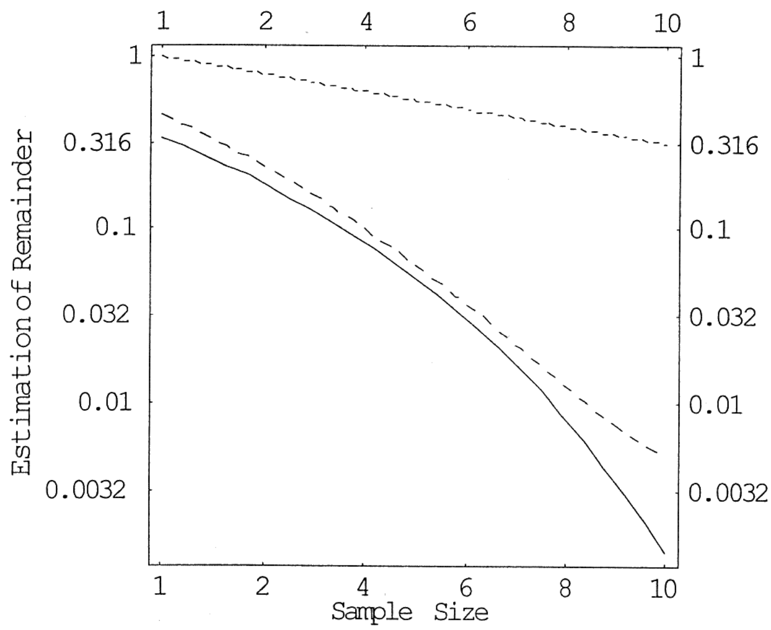
However, for the special case of the uniform distribution, direct computations (not simulations) of remainders for the cases $r = 4, 6, 8, 10$ using the well-known density function of sample mean, show that the constants associated with higher-order remainders are realized at larger sample sizes, but are not exploding as dramatically large as mentioned above (for r larger than 10, the author was unable to compute them, partly because numerical computations become unstable before the realizations of those constants).

Finally, it is worthwhile to note that, as Ghosh (1994, p. 7) mentioned, "In statistical applications of higher-order asymptotics we never need to go beyond $r = 4$."



In the graph above, both x-axis and y-axis are in log-scale. The dotted, dashed, and solid lines show, respectively, the Berry-Esséen, Uspensky-Sherman, and our improved bound given by (2.13) and (2.14).

In the graph below, we enlarge the first graph for the sample size, $1 \leq n \leq 10$, using y-axis in log-scale.



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