On the solutions of the biconfluent Heun equations

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Abstract

A biconfluent Heun differential equation,

$$xu"(x) + (1 + \alpha - \beta x - 2x^2)u'(x) + \{(\gamma - \alpha - 2)x - \frac{1}{2}(\delta + (\alpha + 1)\beta)\}u(x) = 0$$

in which $(\alpha, \beta, \gamma, \delta) \in \mathbb{C}^4$ has two singular points, 0 and ∞. The singularity is regular at 0 and irregular at ∞. By using k-summability ($k = 2$) we obtain new integral formulas for bases of solutions near ∞. We express the Stokes and central connection coefficients in terms of one of them, denoted by l_{11}. By using the symmetries of the biconfluent Heun equations we obtain functional relations satisfied by l_{11} and we determine one of them which implies the others.

1 Introduction

A biconfluent Heun equation, denoted by $BHE(\alpha, \beta, \gamma, \delta)$, is the equation

$$xu"(x) + (1 + \alpha - \beta x - 2x^2)u'(x) + \{(\gamma - \alpha - 2)x - \frac{1}{2}(\delta + (\alpha + 1)\beta)\}u(x) = 0 \quad (1)$$

in which $(\alpha, \beta, \gamma, \delta) \in \mathbb{C}^4$. Its singular points are 0 and ∞. The singularity is regular at 0 and irregular at ∞.

Received by the editors February 2000 - In revised form : March 2001.
Communicated by A. Bultheel.
1991 *Mathematics Subject Classification :* 30B10, 30E15, 32D15, 34A20, 34A25, 34A30, 34E05, 34B05, 40A05, 40G99, 44A10, 65B10.
Key words and phrases : Ordinary differential equations, Heun's equations, irregular singularity, k-summability, Stokes coefficients, connection coefficients, formal solutions.

It seems that the BHE, often studied, particularly by P. Maroni in [5], have not been studied from the k-summability point of view. A. Duval has adopted this point of view in [2] to study the triconfluent Heun equations and we have done the same in [8] for the double confluent Heun equations. Thanks to this point of view we obtain, in this paper, for the BHE, new integral formulas for solutions in the vicinity of ∞ and we give some new relations satisfied by Stokes and connection coefficients concerning these functions.

The formal power series parts of the formal solutions at ∞ are 2-summable in the sense of [1]. According to the general definition, this means that their Borel transforms can be analytically continued along any direction $-d$, such that d is not a singular direction, in functions of exponential growth at most 2 at ∞. We show that these analytic continuations have a *moderate* growth at ∞. Thus, the integral formulas for the 2-sums are available in sectors of infinite radius.

We give a parametric definition of the symmetries (introduced by P. Maroni in [5]) of the $BHE(\alpha, \beta, \gamma, \delta)$ family. Owing to these symmetries, we get representations of every Stokes and connection coefficients in terms of one of them, l_{11}. Then, we obtain functional relations satisfied by l_{11} and we determine one of them which implies the others and we prove that we can construct a family of possible Stokes and connection matrices with any function which satisfies the above relation.

Notations

We shall denote by Ω the Riemann surface of the Logarithm, by $\tilde{e}^{\theta i}$ the element of Ω with modulus 1 and argument $\theta \in \mathbb{R}$. Thus $\tilde{e}^{\theta i} \neq \tilde{e}^{\theta' i}$ if $\theta \neq \theta'$. For $x \in \Omega$ and $a \in \mathbb{C}$, x^a will denote the complex number $e^{a(\ln x + i \arg x)}$. The projection of x onto $\mathbb{C}^*$ will also be denoted by x, the context will allow us to avoid confusion. Let $d \in \mathbb{R}$, if g is a function defined in the subset $\{re^{id},\ r > 0\}$ of $\mathbb{C}$, we denote by $\int_0^{\infty(d)} g(t)dt^2$ the integral $\int_0^{+\infty} g(re^{id})2re^{2id}dr$. We define

$S(d, \omega, r) = \{x \in \Omega,\ d - \frac{\omega}{2} < \arg x < d + \frac{\omega}{2} \text{ and } |x| > r\}$,

$\overline{S}(d, \omega, r) = \{x \in \Omega,\ d - \frac{\omega}{2} \leq \arg x \leq d + \frac{\omega}{2} \text{ and } |x| \geq r\}$,

$I(d, \omega) = \{\xi \in \mathbb{R},\ d - \frac{\omega}{2} < \xi < d + \frac{\omega}{2}\}$.

Finally, for $a \in \mathbb{C}$, we will denote by $\mathbb{C} - [a, a\,\infty[$ the set of the complex numbers which are not in the set $\{re^{i \arg a} \ /\ r \geq |a|\}$.

2 Transformations of $BHE(\alpha, \beta, \gamma, \delta)$:

Let $\tilde{\Lambda}$ be the set of maps from $\mathbb{C}^4$ into $\mathbb{C}$. Let Λ be the vector space over $\mathbb{C}$ of the maps $y : \mathbb{C}^4 \times \Omega \longrightarrow \mathbb{C}$ such that, for $(\alpha, \beta, \gamma, \delta) \in \mathbb{C}^4$, $y(\alpha, \beta, \gamma, \delta;\ )$ is a solution of $BHE(\alpha, \beta, \gamma, \delta)$.

Taking proposition 1.1.1 of [5] into account, we can state the following definition of the symmetries of BHE.

Definition 1. *Let* $(\epsilon, h) \in \{-1, 1\} \times \mathbb{Z}$. *We denote by* $t_{\epsilon,h}$ *the map from* Λ *into itself defined for* $y \in \Lambda$ *by*

$$t_{\epsilon,h}(y)(\alpha,\beta,\gamma,\delta;x) = x^{\frac{\epsilon-1}{2}\alpha} e^{\frac{1-(-1)^h}{2}(x^2+\beta x)} y(\epsilon\alpha, i^h\beta, (-1)^h\gamma, (-i)^h\delta; \tilde{e}^{h\frac{\pi}{2}i}x).$$

We denote by T *the set* $\{t_{\epsilon,h} \ / \ (\epsilon, h) \in \{-1, 1\} \times \mathbb{Z}\}$.

Particularly we have: $t_{1,1}(y)(\alpha,\beta,\gamma,\delta;x) = e^{x^2+\beta x} y(\alpha, i\beta, -\gamma, -i\delta; \tilde{e}^{\frac{\pi}{2}i}x)$ and $t_{-1,0}(y)(\alpha,\beta,\gamma,\delta;x) = x^{-\alpha} y(-\alpha,\beta,\gamma,\delta;x)$.

Definition 2. *Let* $(\epsilon, h) \in \{-1, 1\} \times \mathbb{Z}$, $l \in \tilde{\Lambda}$ *and* $y \in \Lambda$. *We define the element* ly *of* Λ, *the map* $\tilde{t}_{\epsilon,h}$ *from* $\tilde{\Lambda}$ *into itself, the set* $\tilde{T}$ *and the map* $lt_{\epsilon,h}$ *from* Λ *to* Λ *by*

1. $ly(\alpha,\beta,\gamma,\delta;x) = l(\alpha,\beta,\gamma,\delta) y(\alpha,\beta,\gamma,\delta;x)$
2. $\tilde{t}_{\epsilon,h}(l)(\alpha,\beta,\gamma,\delta) = l(\epsilon\alpha, i^h\beta, (-1)^h\gamma, (-i)^h\delta)$
3. $\tilde{T} = \{\tilde{t}_{\epsilon,h} \ / \ (\epsilon, h) \in \{-1, 1\} \times \boldsymbol{Z}\}$
4. $(lt_{\epsilon,h})(y) = l(t_{\epsilon,h}(y))$

One easily verifies the two next propositions.

Proposition 3. 1. $\tilde{T} = \{\tilde{t}_{\epsilon,h} \ / \ (\epsilon, h) \in \{-1, 1\} \times \{-1, 0, 1, 2\}\}$.

2. *For* (ϵ, h), $(\epsilon', h') \in \{-1, 1\} \times \mathbb{Z}$, $\tilde{t}_{\epsilon',h'} \circ \tilde{t}_{\epsilon,h} = \tilde{t}_{\epsilon'\epsilon,h'+h}$.
3. $(\tilde{T}, \circ)$ *is a group isomorphic to* $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

Proposition 4. *Let* $(\epsilon, h) \in \{-1, 1\} \times \mathbb{Z}$. *Then*

1. $t_{\epsilon,h}$ *is a linear map from* Λ *into itself.*
2. *If* $(y, l) \in \Lambda \times \tilde{\Lambda}$ *then* $t_{\epsilon,h}(l\,y) = \tilde{t}_{\epsilon,h}(l)\, t_{\epsilon,h}(y)$.
3. *For all* (ϵ, h), $(\epsilon', h') \in \{-1, 1\} \times \mathbb{Z}$, $t_{\epsilon',h'} \circ t_{\epsilon,h} = e^{ih'\frac{\pi}{2}\frac{\epsilon-1}{2}\epsilon'\alpha} t_{\epsilon'\epsilon,h'+h}$

According to item 3 of the above proposition, T is not a group. However we have

$$\begin{aligned} t_{\epsilon',h'} \circ t_{1,h} &= t_{\epsilon',h+h'} \\ t_{-1,0} \circ t_{\epsilon,h} &= t_{-\epsilon,h} \end{aligned} \tag{2}$$

and, for any $t \in T$, there exists a unique $(\epsilon, j, h) \in \{-1, 1\} \times \{0, 1, -1, 2\} \times \mathbb{Z}$ such that $t = t_{\epsilon,j} \circ t_{1,4h}$.

From now on we shall denote $t't$ instead of $t' \circ t$.

2.1 Solutions at 0

P. Maroni details in [5] the solutions at 0 which is a regular singularity. If $\alpha \in \mathbb{Z}$, then, except for some values, there are logarithmic terms in the bases of solutions. We do not study in this paper the analyticity of the functions in respect to the parameters $\alpha, \beta, \gamma, \delta$.

Proposition 5. *Let* $(\alpha, \beta, \gamma, \delta) \in (\mathbb{C} - \mathbb{Z}) \times \mathbb{C}^3$ *and*

$$N(\alpha, \beta, \gamma, \delta; x) = \Gamma(\alpha) \sum_{n=0}^{+\infty} \frac{A_n(\alpha, \beta, \gamma, \delta)}{\Gamma(\alpha + 1 + n)n!} x^n$$

where the A_n *are polynomials in* α, β, γ, δ *defined by the following relation*

$$A_{n+2} = \{\beta(n+1) + \tfrac{1}{2}(\delta + \beta(1+\alpha)\}A_{n+1} - (\gamma - 2 - \alpha - 2n)(n+1)(n+1+\alpha)A_n$$
for $n \geq 0$ *and* $A_{-1} = 0,\ A_0 = 1$.

Then $(N(\alpha, \beta, \gamma, \delta; x), t_{-1,0}N(\alpha, \beta, \gamma, \delta; x))$ *is a basis of solutions of* $BHE(\alpha, \beta, \gamma, \delta)$.

The function $N(\alpha, \beta, \gamma, \delta; x)$ is an entire function in x.

If we denote by M the function $t_{-1,0}N$, we have

Proposition 6. *Let* $\alpha \notin \mathbb{Z}$ *and* $(\epsilon, h) \in \{-1, 1\} \times \mathbb{Z}$. *One has*

1. $t_{1,h}(N) = N$, $t_{1,h}(M) = e^{-h\alpha\frac{\pi}{2}i}M$,
2. $t_{-1,0}(M) = N$

3 Solutions at ∞ of the $BHE(\alpha, \beta, \gamma, \delta)$

Definition 7. *Let* d, ω, r *be real numbers with* $\omega > 0$, $r \geq 0$. *An analytic function* f *in a sector* $S(d, \omega, r)$ *has a moderate growth at* ∞ *in* $S(d, \omega, r)$ *if to all sector* $\overline{S}(d, \omega - \varepsilon, \rho)$ *with* $0 \leq \varepsilon < \omega$ *and* $\rho > r$ *there exists* $A > 0$, $\lambda \in \mathbb{R}$ *such that for each* $x \in \overline{S}(d, \omega - \varepsilon, \rho)$, $|f(x)| \leq A|x|^\lambda$.

We will use some definitions and results about 2-summability we can find in [1]. In this section, we first give a basis of formal solutions of BHE at ∞, the irregular singularity. Owing to 2-summability we obtain a family of actual solutions which admit these formal solutions as asymptotic expansions in proper sectors and for which we give integral representations available in $S(d, \frac{\pi}{2}, 0)$ where d is not a singular direction.

Proposition 8. *Let* $(\alpha, \beta, \gamma, \delta) \in \mathbb{C}^4$. *Then* $BHE(\alpha, \beta, \gamma, \delta)$ *admits as basis of formal solutions at* ∞, $(\widehat{y}_0(\alpha, \beta, \gamma, \delta; x), \widehat{y}_1(\alpha, \beta, \gamma, \delta; x))$ *with*

1. $\widehat{y}_0(\alpha,\beta,\gamma,\delta;x) = x^{\frac{1}{2}(\gamma-\alpha-2)}\widehat{s}_0(\alpha,\beta,\gamma,\delta;x)$,
2. $\widehat{y}_1(\alpha,\beta,\gamma,\delta;x) = x^{-\frac{1}{2}(\gamma+\alpha+2)}e^{x^2+\beta x}\widehat{s}_1(\alpha,\beta,\gamma,\delta;x)$
3. $\widehat{s}_0(\alpha,\beta,\gamma,\delta;x) = \sum_{n=0}^{+\infty} a_n(\alpha,\beta,\gamma,\delta)x^{-n}$, *where the complex numbers* a_n *are defined by:*

$$2(n+2)a_{n+2} = \{\frac{1}{2}(\delta+\beta(\gamma-1))-\beta(n+1)\}a_{n+1}-(\frac{\gamma-\alpha-2}{2}-n)(\frac{\gamma+\alpha-2}{2}-n)a_n$$

for $n \geq 0$ *and* $a_{-1} = 0, a_0 = 1$

4. $\widehat{s}_1(\alpha,\beta,\gamma,\delta;x) = \widehat{s}_0(\alpha,i\beta,-\gamma,-i\delta;ix)$

Proof : By direct computation (*c.f* [14]), using the fact that the Newton polygon at ∞ of BHE$(\alpha,\beta,\gamma,\delta)$ has one slope equal to 0 and one slope equal to 2, we obtain the formal solutions.

In order to prove the moderate growth of the Borel transform $\widehat{\mathcal{B}}_2\widehat{s}_i$ we need the two following lemmas. The first one is a particular case of a general result recalled in [6] (page 88).

Lemma 9. *If the coefficients of a formal power series* $\widehat{f} = \sum_{n=0}^{+\infty} a_n x^n$ *satisfy the recurrence relation* $P_0(n)a_n + P_1(n)a_{n+1} + P_2(n)a_{n+2} = 0$ *where* P_i *is the polynomial of* $\mathbb{C}[x]$ *defined by* $P_i(x) = \sum_{l=0}^{p} \alpha_{i,l}x^l$ *and* $(i,p) \in \{0,1,2\}\times\mathbb{N}$ *then* $\widehat{f}$ *is a solution of the differential equation*

$$\sum_{i=0}^{2}\sum_{l=0}^{p}\alpha_{i,l}(x\frac{d}{dx})^l x^{-i}\widehat{f} = (a_0P_1(-1)+a_1P_2(-1))x^{-1}+a_0P_2(-2)x^{-2}.$$

Lemma 10. *Let* $a \in \mathbb{C}$ *such that* $Re(a) < 1$. *Let* $d,\omega \in \mathbb{R}\times\mathbb{R}^+$. *Let* $\sum_{n=0}^{\infty} a_n x^n$ *be a convergent series that can be analytically continued to a function g(x) with a moderate growth at* ∞ *in* $S(d,\omega,0)$. *Then the series* $\sum_{n=0}^{\infty} a_n \frac{\Gamma(n+1)}{\Gamma(n+1-a)}x^n$ *is convergent and can be analytically continued to a function with a moderate growth at* ∞ *in* $S(d,\omega,0)$.

Proof : The convolution $(x^{-a} * g)(t) = \frac{d}{dt}\int_0^t (t-u)^{-a}g(u)du$ is defined in $S(d,\omega,0)$. Let R be the radius of convergence of the series $\sum_{n=0}^{\infty} a_n x^n$. Then, for $|t| < R$,

$$\int_0^t (t-u)^{-a}g(u)du = \int_0^t (t-u)^{-a}\sum_{n=0}^{\infty} a_n u^n du = \sum_{n=0}^{\infty} a_n \int_0^t (t-u)^{-a}u^n du =$$

$$t^{-a+1}\sum_{n=0}^{\infty} a_n(\int_0^1 (1-\theta)^{-a}\theta^n d\theta)t^n = t^{-a+1}\sum_{n=0}^{\infty} a_n\frac{\Gamma(-a+1)\Gamma(n+1)}{\Gamma(-a+1+n+1)}t^n$$

and

$$(x^{-a} * g)(t) = \frac{d}{dt}(t^{-a+1}\sum_{n=0}^{\infty} a_n \frac{\Gamma(-a+1)\Gamma(n+1)}{\Gamma(-a+1+n+1)}t^n) = t^{-a}\sum_{n=0}^{\infty} a_n \frac{\Gamma(n+1)}{\Gamma(n+1-a)}t^n.$$

The classical properties of convolution and the hypotheses imply that $t^a(x^{-a} * g)(t)$ is analytic in $S(d,\omega,0)$ and has a moderate growth at ∞ in $S(d,\omega,0)$. Since $t^a(x^{-a} * g)(t) = \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+1)}{\Gamma(n+1-a)}t^n$ in the disc of convergence of $\sum_{n=0}^{\infty} a_n x^n$, $t^a(x^{-a} * g)(t)$ is the analytic continuation of $\sum_{n=0}^{\infty} a_n \frac{\Gamma(n+1)}{\Gamma(n+1-a)}t^n$ in $S(d,\omega,0)$.

Proposition 11. *Let $(\alpha,\beta,\gamma,\delta) \in \mathbb{C}^4$ and $j \in \{0,1\}$ Then*

1. *The formal power series $\widehat{s}_j(\alpha,\beta,\gamma,\delta;x)$ is 2-summable at ∞ in every direction $d \neq (2k+1+j)\frac{\pi}{2}$ with $k \in \mathbb{Z}$.*

2. *The Borel transform of order 2, $\widehat{\mathcal{B}}_2\widehat{s}_j(t)$ is convergent and can be analytically continued in*
   $\mathbb{C} - ([i^{j+1}, i^{j+1}\,\infty[\cup [-i^{j+1}, -i^{j+1}\,\infty[)$ to a function $g_j(\alpha,\beta,\gamma,\delta;t)$ which has a moderate growth at ∞ in every direction different from $(2k+1+j)\frac{\pi}{2}$,

3. *If $d \neq (2k+1+j)\frac{\pi}{2}$ the 2-sum of $\widehat{s}_j(\alpha,\beta,\gamma,\delta;x)$ in direction d is given for all x in sector $S(d,\frac{\pi}{2},0)$ by*

$$s_{j,d}(\alpha,\beta,\gamma,\delta;x) = x^2\int_0^{\infty(-d)} g_j(\alpha,\beta,\gamma,\delta;t)e^{-t^2x^2}dt^2 \tag{3}$$

Proof : According to the general properties of formal solutions of differential equations (*c.f* [12]), $\widehat{s}_0(\alpha,\beta,\gamma;x)$ is 2-summable in every direction that differs from $(2k+1)\frac{\pi}{2}$, the directions of maximal decrease of $e^{x^2+\beta x}$. Classic results about k-summability (*c.f* [1]) prove that the exponential growth near ∞ of the Borel transform is at most 2. Let us prove that the growth is moderate. When n is odd, the factor $\Gamma(1+\frac{n}{2})$ does not allow us to use lemma 9. In order to get around this difficulty we write

$$\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(1+\frac{n}{2})}t^n = \sum_{n=0}^{\infty}\frac{a_{2n}}{\Gamma(1+n)}t^{2n} + t\sum_{n=0}^{\infty}\frac{a_{2n+1}}{\Gamma(1+n+\frac{1}{2})}t^{2n}. \tag{4}$$

From the recurrence relation given by item 3 of proposition 8 we deduce recurrence relations satisfied by the coefficients a_{2n}, a_{2n+1} and then recurrence relations satisfied by $\frac{a_{2n}}{\Gamma(n+1)}$ and $\frac{a_{2n+1}}{\Gamma(n+1)}$. Finally, by using lemma 9, we obtain differential equations satisfied by $\sum_{n=0}^{\infty}\frac{a_{2n}}{\Gamma(n+1)}t^n$ and $\sum_{n=0}^{\infty}\frac{a_{2n+1}}{\Gamma(n+1)}t^n$. We give them in the appendix. These equations have been calculated by programming with mathematica 3.0.

1. If $\beta \neq 0$, the form of these equations is $\sum_{i=0}^{4} p_i(x)z^{(i)} + \beta x^3(1+x)^2 z^{(5)}(x) = 0$ where, for all i, $p_i(x)$ is a polynomial in x the degree of which is less or equal to i. The singularities are 0, -1, ∞. The singularity at ∞ is regular.

2. If $\beta = 0$ and $\delta \neq 0$, we obtain the form $\sum_{i=0}^{3} p_i(x)z^{(i)} + x^2(1+x)^2 z^{(4)}(x) = 0$ with the same characteristics as above.

   The above remarks show that, if $(\beta,\delta) \neq (0,0)$, the sums of $\sum_{n=0}^{\infty} \frac{a_{2n}}{\Gamma(n+1)} t^n$ and $\sum_{n=0}^{\infty} \frac{a_{2n+1}}{\Gamma(n+1)} t^n$ can be analytically continued in $\mathbb{C} -]-1,-\infty[$ with a moderate growth at ∞. According to lemma 10, $\sum_{n=0}^{\infty} \frac{a_{2n+1}}{\Gamma(1+n+\frac{1}{2})} t^n$ is convergent and can be analytically continued in $\mathbb{C} -]-1,-\infty[$ with a moderate growth at ∞. Using (4) we conclude the proof of item 2 of the proposition for $j = 0$. Since the growth is moderate, the Laplace transform (3) is defined in the sector $S(d, \frac{\pi}{2}, 0)$. For $\widehat{s}_1$, we can use item 4 of proposition 8.

3. If $\beta = \delta = 0$ we know (*c.f* [5] page 195) that the solutions y of $BHE(\alpha,0,\gamma,0)$ are the functions $y(x) = z(x^2)$ where $z(t)$ is a solution of the Kummer's equation $K(a,c) : tz''(t) + (c-t)z'(t) - az(t) = 0$ in which $a = \frac{\alpha-\gamma+2}{4}$, $c = \frac{\alpha}{2}+1$. Then, $\widehat{\mathcal{B}}_2 \widehat{s}_0(\alpha,0,\gamma,0;t) = {}_2F_1(a, a+1-c, 1; -t^2)$ and $\widehat{\mathcal{B}}_2 \widehat{s}_1(\alpha,0,\gamma,0;t) = {}_2F_1(c-a, 1-a, 1; t^2)$ where ${}_2F_1(a,b,1;z)$ denotes the hypergeometric series $\sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{n! n!}$. This series is a solution of the hypergeometric equation $E(a,b,1)$: $x(1-x)z''(x) - (x(a+b+1)-1)z'(x) - abz(x) = 0$. The singularities of $E(a,b,1)$, $0, 1, \infty$ are regular. Hence, the convergent series ${}_2F_1(a,b,1;z)$ has an analytic continuation, denoted by ${}_2f_1(a,b,1;z)$, in $\mathbb{C} -]1,+\infty[$ with moderate growth at ∞. We conclude that

   for $d \neq (2k+1)\frac{\pi}{2}$ and $x \in S(d, \frac{\pi}{4}, 0)$,

$$s_{0,d}(\alpha,0,\gamma,0) = x^2 \int_0^{\infty(-d)} {}_2f_1(a, a+1-c, 1; -t^2) e^{-t^2 x^2} dt^2,$$

   for $d \neq k\pi$ and $x \in S(d, \frac{\pi}{4}, 0)$,

$$s_{1,d}(\alpha,0,\gamma,0) = x^2 \int_0^{\infty(-d)} {}_2f_1(c-a, 1-a, 1; t^2) e^{-t^2 x^2} dt^2$$

Let $(\alpha,\beta,\gamma,\delta,k) \in \mathbb{C}^4 \times \mathbb{Z}$ and $j = \frac{1-(-1)^k}{2}$. Classic results (*c.f* [1]) show that there exists one analytic function in Ω, denoted by $s_k(\alpha,\beta,\gamma,\delta;x)$, such that, for any $d \in I(k\frac{\pi}{2}, \pi)$ and any $x \in S(d, \frac{\pi}{2}, 0)$,

$$s_k(\alpha,\beta,\gamma,\delta;x) = s_{j,d}(\alpha,\beta,\gamma,\delta;x).$$

We denote by $s_k(\alpha,\beta,\gamma,\delta;x)$ this analytic continuation. One can easily prove the next proposition.

Proposition 12. *Let* $(\alpha,\beta,\gamma,\delta,k) \in \mathbb{C}^4 \times \mathbb{Z}$ *and* $j = \frac{1-(-1)^k}{2}$. *For each* $\varepsilon \in \left]0, \frac{3\pi}{2}\right[$ *there exists a real number* $r > 0$ *such that* $s_k(\alpha,\beta,\gamma,\delta;x)$ *is 2-Gevrey asymptotic to* $\widehat{s}_j(\alpha,\beta,\gamma,\delta;x)$ *in sector* $S(k\frac{\pi}{2}, \frac{3\pi}{2} - \varepsilon, r)$.

Definition 13. *Let* $(\alpha,\beta,\gamma,\delta,k) \in \mathbb{C}^4 \times \mathbb{Z}$. *We define, for* $x \in \Omega$,

$$y_{2k}(\alpha,\beta,\gamma,\delta;x) = x^{\frac{1}{2}(\gamma-\alpha-2)} s_{2k}(\alpha,\beta,\gamma,\delta;x) \tag{5}$$

$$y_{2k+1}(\alpha,\beta,\gamma,\delta;x) = x^{-\frac{1}{2}(\gamma+\alpha+2)} s_{2k+1}(\alpha,\beta,\gamma,\delta;x)\, e^{x^2+\beta x} \tag{6}$$

Then we have

Proposition 14. *Let* $k \in \mathbb{Z}$. *Let* $j = \frac{1-(-1)^k}{2}$.

- $(y_k(\alpha,\beta,\gamma,\delta;x), y_{k+1}(\alpha,\beta,\gamma,\delta;x))$ *is a basis of solutions of* $BHE(\alpha,\beta,\gamma,\delta)$.
- *Let* $d \in I(k\frac{\pi}{2},\pi)$. *For all* $x \in S(d,\frac{\pi}{2},0)$,

$$y_k(\alpha,\beta,\gamma,\delta;x) = x^{\frac{1}{2}((-1)^k\gamma-\alpha+2)} \int_0^{\infty(-d)} g_j(\alpha,\beta,\gamma,\delta;t)e^{-t^2x^2}dt^2 e^{j(x^2+\beta x)}$$

In order to study the action of the symmetries on the above bases we need two lemmas.

Lemma 15. *Let* $(\alpha,\beta,\gamma,\delta) \in \mathbb{C}^4$ *and* $j \in \{0,1\}$. *We have*

1. $\widehat{s}_j(\alpha,i\beta,-\gamma,-i\delta;ix) = \widehat{s}_{1-j}(\alpha,\beta,\gamma,\delta;x)$
2. $\widehat{s}_j(-\alpha,\beta,\gamma,\delta;x) = \widehat{s}_j(\alpha,\beta,\gamma,\delta;x)$

Proof : Let $(\alpha,\beta,\gamma,\delta) \in \mathbb{C}^4$. We define

$$\widehat{y}_0(\alpha,i\beta,-\gamma,-i\delta;\tilde{e}^{+i\frac{\pi}{2}}x) = e^{i\frac{\pi}{4}(-\gamma-\alpha-2)}x^{\frac{1}{2}(-\gamma-\alpha-2)}\widehat{s}_0(\alpha,i\beta,-\gamma,-i\delta;ix).$$

Since $\widehat{y}_0(\alpha,\beta,\gamma,\delta;x)$ is a solution of $BHE(\alpha,\beta,\gamma,\delta))$, we verify that $e^{x^2+\beta x}\widehat{y}_0(\alpha,i\beta,-\gamma,-i\delta;\tilde{e}^{+i\frac{\pi}{2}}x)$ is also a solution of that equation. Since $\widehat{y}_1$ is the unique formal solution of $BHE(\alpha,\beta,\gamma,\delta)$ of the form $x^{\frac{1}{2}(-\gamma-\alpha-2)}e^{x^2+\beta x}\widehat{s}(x)$ with $\widehat{s}(x) \in \mathbb{C}[[x^{-1}]]$ and having 1 as constant coefficient, we have the first formula of the lemma for $j=0$. In the same way we obtain the other formulae.

Lemma 16. *Let* $(\alpha,\beta,\gamma,\delta,k) \in \mathbb{C}^4 \times \mathbb{Z}$. *We have*

1. $s_k(\alpha,i\beta,-\gamma,-i\delta;\tilde{e}^{i\frac{\pi}{2}}x) = s_{k-1}(\alpha,\beta,\gamma,\delta;x)$,
2. $s_k(-\alpha,\beta,\gamma,\delta;x) = s_k(\alpha,\beta,\gamma,\delta;x)$

Proof : Let $k \in \mathbb{Z}$ and $j = \frac{1-(-1)^k}{2}$. According to proposition 12, the function $s_k(\alpha,i\beta,-\gamma,-i\delta;x)$ is 2-Gevrey asymptotic to $\widehat{s}_j(\alpha,i\beta,-\gamma,-i\delta;x)$ in sector $S(k\frac{\pi}{2},\frac{3\pi}{2}-\varepsilon,r)$. Then $s_k(\alpha,i\beta,-\gamma,-i\delta;\tilde{e}^{i\frac{\pi}{2}}x)$ is 2-Gevrey asymptotic to $\widehat{s}_j(\alpha,i\beta,-\gamma,-i\delta;ix)$ for x being in sector $S((k-1)\frac{\pi}{2},\frac{3\pi}{2}-\varepsilon,r)$. Since $\widehat{s}_j(\alpha,i\beta,-\gamma,-i\delta;ix) = \widehat{s}_{1-j}(\alpha,\beta,\gamma,\delta;x)$, $s_k(\alpha,i\beta,-\gamma,-i\delta;\tilde{e}^{i\frac{\pi}{2}}x)$ is the 2-sum of $\widehat{s}_{1-j}(\alpha,\beta,\gamma,\delta;x)$ in sector $S((k-1)\frac{\pi}{2},\frac{3\pi}{2}-\varepsilon,r)$. The unicity of the 2-sum implies the first result of the proposition. We can prove the second one in the same way.

Proposition 17. *For each $k \in \mathbb{Z}$,*

$$\begin{aligned} t_{1,1}(y_k) &= -ie^{((-1)^{k+1}\gamma-\alpha)\frac{\pi}{4}i}y_{k-1} \\ t_{-1,0}(y_k) &= y_k \end{aligned} \tag{7}$$

Proof : The following formula

$$t_{1,1}y_{2k}(\alpha,\beta,\gamma,\delta;x) = e^{x^2+\beta x}(-i)e^{(-\gamma-\alpha)\frac{\pi}{4}i}x^{\frac{1}{2}(-\gamma-\alpha-2)}s_{2k}(\alpha,i\beta,-\gamma,-i\delta,\tilde{e}^{i\frac{\pi}{2}}x)$$

and lemme 16 imply the first result for the even indices. Similarly, we can prove the other results.

Proposition 18. *For all $(\varepsilon,h,k) \in \{-1,1\} \times \mathbb{Z} \times \mathbb{Z}$,*

$$t_{\varepsilon,h}y_k = (-i)^h e^{h((-1)^{h+k}\gamma-\varepsilon\alpha)\frac{\pi}{4}i}y_{k-h} \tag{8}$$

Proof : Relation (7) is the formula (8) for $\varepsilon = 1,\ h = 1$. By applying $t_{1,-1}$ to (7) and using item 2 of the proposition 4 we obtain relation (8) for $\varepsilon = 1, h = -1$. By induction we can prove the relation for $\varepsilon = 1, h \in \mathbb{Z}$. Finally we apply $t_{-1,0}$ to the above relation and we obtain (8).

Connection to the solutions defined by P. Maroni

We are now going to compare the functions B^+, B^-, ... defined by P. Maroni on subsets of $\mathbb{C}$ with our functions y_i defined on Ω.

Let us assume that, for $x \in \mathbb{C}$ such that $Re(x) > 0$, $x^a = e^{a\ln|x|+i\arg x}$ where $\arg x \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[$ and let $\tilde{x}$ be the element of Ω with $\arg\tilde{x} \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[$ and whose projection onto $\mathbb{C}^*$ is x.

According to proposition 12, the function $y_0(\tilde{x})$ is asymptotic at infinity to $\widehat{y_0}$ in sectors $S(0,\frac{3\pi}{2}-\varepsilon,r)$. Thus, the definition of $B^+(\alpha,\beta,\gamma,\delta;x)$ in [5], page 210 and unicity given by proposition 3.4.1 in [5] imply

$$y_0(\tilde{x}) = B^+(x),\ Re(x) > 0,\ \arg\tilde{x} \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[\tag{9}$$

Owing to proposition 18 and the definitions of B^-, E^+ and E^-, we can easily obtain the following identities

$$\begin{aligned} y_1(\tilde{x}) &= -ie^{-(\gamma+\alpha)\frac{\pi}{4}i}E^+(x), & Im(x) > 0,\ & \arg\tilde{x} \in \left]0,\pi\right[\\ y_{-2}(\tilde{x}) &= -e^{(-\gamma+\alpha)\frac{\pi}{2}i}B^-(x), & Re(x) < 0,\ & \arg\tilde{x} \in \left]-\frac{\pi}{2},-\frac{3\pi}{2}\right[\\ y_{-1}(\tilde{x}) &= ie^{(\gamma+\alpha)\frac{\pi}{4}i}E^-(x), & Im(x) < 0,\ & \arg\tilde{x} \in \left]-\pi,0\right[\end{aligned} \tag{10}$$

On the other hand, taking the sectors in which $y_{-1}(x)$ and $y_1(x)$ are asymptotic to $\widehat{y}_1(x)$ at ∞ into account, it seems that those functions are not equal to $H^+(x)$. We need a more detailed study to make clear the connection between $y_i(x)$ and $H^+(x)$.

4 Stokes and connection coefficients

As two bases of solutions of $BHE(\alpha,\beta,\gamma,\delta)$ are connected by an invertible constant matrix we can give the following definitions of Stokes and connection matrices.

Definition 19. *Let* $(\alpha,\beta,\gamma,\delta,k)\in\mathbb{C}^4\times\mathbb{Z}$. *Let* $\mathcal{M}_k(\alpha,\beta,\gamma,\delta)$ *such that, for all* $x\in\Omega$,

$$(y_{2k}(\alpha,\beta,\gamma,\delta;x),y_{2k-1}(\alpha,\beta,\gamma,\delta;x))=$$
$$(y_{2k}(\alpha,\beta,\gamma,\delta;x),y_{2k+1}(\alpha,\beta,\gamma,\delta;x))\mathcal{M}_{2k}(\alpha,\beta,\gamma,\delta)$$
$$(y_{2k}(\alpha,\beta,\gamma,\delta;x),y_{2k+1}(\alpha,\beta,\gamma,\delta;x))=$$
$$(y_{2k+2}(\alpha,\beta,\gamma,\delta;x),y_{2k+1}(\alpha,\beta,\gamma,\delta;x))\mathcal{M}_{2k+1}(\alpha,\beta,\gamma,\delta)$$

These matrices are called Stokes matrices.

Definition 20. *Let* $(\alpha,\beta,\gamma,\delta)\in(\mathbb{C}-\mathbb{Z})\times\mathbb{C}^3$, *we call connection matrix of* $BHE(\alpha,\beta,\gamma,\delta)$ *the matrix* $\mathcal{L}(\alpha,\beta,\gamma,\delta)$ *defined by the following relation, available for all* $x\in\Omega$,

$$(N(\alpha,\beta,\gamma,\delta;x),M(\alpha,\beta,\gamma,\delta;x))=(y_0(\alpha,\beta,\gamma,\delta;x),y_{-1}(\alpha,\beta,\gamma,\delta;x))\mathcal{L}(\alpha,\beta,\gamma,\delta) \tag{11}$$

The connection coefficients are the complexe numbers $l_{ij}(\alpha,\beta,\gamma,\delta)$ *defined by* $\mathcal{L}(\alpha,\beta,\gamma,\delta)=(l_{ij}(\alpha,\beta,\gamma,\delta))_{i,j\in\{1,2\}}$

Proposition 21. *Let* $(\alpha,\beta,\gamma,\delta,k)\in\mathbb{C}^4\times\mathbb{Z}$.

$$\mathcal{M}_{2k}(\alpha,\beta,\gamma,\delta)=\begin{pmatrix}1 & c_{2k}(\alpha,\beta,\gamma,\delta)\\ 0 & 1\end{pmatrix}\quad \mathcal{M}_{2k+1}(\alpha,\beta,\gamma,\delta)=\begin{pmatrix}1 & 0\\ c_{2k+1}(\alpha,\beta,\gamma,\delta) & 1\end{pmatrix} \tag{12}$$

Complex numbers $c_k(\alpha,\beta,\gamma,\delta)$ *are called Stokes coefficients.*

Proof : Let $(\alpha,\beta,\gamma,\delta,k)\in\mathbb{C}^4\times\mathbb{Z}$. Let a, $c_{2k+1}\in\mathbb{C}$ such that, for all $x\in\Omega$, $y_{2k}(\alpha,\beta,\gamma,\delta;x)=ay_{2k+2}(\alpha,\beta,\gamma,\delta;x)+c_{2k+1}y_{2k+1}(\alpha,\beta,\gamma,\delta;x)$. Then we have $s_{2k}=as_{2k+2}+c_{2k+1}x^{-\gamma}s_{2k+1}e^{x^2+\beta x}$. According to proposition 12, there exists a sector $S((2k+1)\frac{\pi}{2},\frac{\pi}{4},r)$ where $s_{2k}(x)$, $s_{2k+2}(x)$ (respectively $s_{2k+1}(x)$) have, as aymptotic expansion, $\widehat{s}_0^\infty$ (respectively $\widehat{s}_1$). If $x\to\infty$ in this sector, we obtain $a=1$.

In the same way we obtain $\mathcal{M}_{2k}$.

4.1 Functional relations satisfied by Stokes and connection coefficients

By applying transformations $t_{\varepsilon,h}$ to $y_{k-1} = c_k y_k + y_{k+1}$ we obtain

Proposition 22. *For all* $(\varepsilon, h, k) \in \{-1, 1\} \times \mathbb{Z} \times \mathbb{Z}$,

$$c_{k-h} = e^{(-1)^{k+h} h\gamma \frac{\pi}{2} i} \tilde{t}_{\varepsilon,h} c_k \tag{13}$$

$$c_k = e^{(-1)^{k-1} \gamma k \frac{\pi}{2} i} \tilde{t}_{1,-k} c_0 \tag{14}$$

$$c_k = \tilde{t}_{-1,0} c_k \tag{15}$$

So, every Stokes coefficient can be expressed in terms of c_0. Relation (13) is equivalent to the two relations (14) and (15).

Let us denote by $W(y_1(x), y_2(x))$ the wronskian of two solutions $y_1(x), y_2(x)$ of $BHE(\alpha, \beta, \gamma, \delta)$.

Proposition 23. *For* $(\alpha, \beta, \gamma, \delta, k, \varepsilon) \in \mathbb{C}^4 \times \mathbb{Z} \times \{-1, 1\}$, *we have*

$$W(y_{2k}(\alpha, \beta, \gamma, \delta; x), y_{2k+\varepsilon}(\alpha, \beta, \gamma, \delta; x)) = 2x^{-(\alpha+1)} e^{\beta x + x^2} \tag{16}$$

$$W(N(\alpha, \beta, \gamma, \delta; x), M(\alpha, \beta, \gamma, \delta; x) = -\alpha x^{-(\alpha+1)} e^{\beta x + x^2} \tag{17}$$

Proof : The wronskian of two solutions of $BHE(\alpha, \beta, \gamma, \delta)$ is a solution of the differential equation $xW'(x) + (1 + \alpha - \beta x - 2x^2)W(x) = 0$. Thus it is of the form $Kx^{-(\alpha+1)} e^{\beta x + x^2}$, in which K is a complex number independent of x. By definition of y_{2k}, y_{2k-1}, $W(y_{2k}, y_{2k-1}) = x^{-(\alpha+1)} e^{\beta x + x^2} f(x)$ with

$$f(x) = 2s_{2k}s_{2k-1} - x^{-2}(s_{2k})' s_{2k-1} + (-\gamma x^{-2} + \beta x^{-1}) s_{2k} s_{2k-1} + x^{-1} s_{2k} (s_{2k-1})'$$

According to proposition 12 there exists a sector $S(2k\frac{\pi}{2}, \frac{\pi}{4}, r)$ in which $f(x)$ admits $2\widehat{s}_0\widehat{s}_1 - x^{-2}(\widehat{s}_0)'\widehat{s}_1 + (-\gamma x^{-2} + \beta x^{-1})\widehat{s}_0\widehat{s}_1 + x^{-1}\widehat{s}_0(\widehat{s}_1)'$ as asymptotic expansion. Hence the limit of $f(x)$ when $x \longrightarrow \infty$ in the sector is 2.

Let $t \in T$. One denotes by $M(t, 0)(\alpha, \beta, \gamma, \delta)$ (respectively by $M(t, \infty)(\alpha, \beta, \gamma, \delta)$) the matrix expressing t in the basis $(N(\alpha, \beta, \gamma, \delta; x), M(\alpha, \beta, \gamma, \delta; x))$ (respectively in the basis $(y_0(\alpha, \beta, \gamma, \delta; x), y_{-1}(\alpha, \beta, \gamma, \delta; x))$. So $M(t, 0)$ and $M(t, \infty)$ are maps from $(\mathbb{C} - \mathbb{Z}) \times \mathbb{C}^3$ into the set of matrices whose coefficients are complex numbers. Proposition 18 and the definition of the Stokes matrices give

$$M(t_{1,-1}, \infty) = \begin{pmatrix} -c_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} ie^{(\gamma+\alpha)\frac{\pi}{4}i} & 0 \\ 0 & ie^{(-\gamma+\alpha)\frac{\pi}{4}i} \end{pmatrix} \tag{18}$$

$$M(t_{1,1}, \infty) = \begin{pmatrix} 0 & 1 \\ 1 & c_{-1} \end{pmatrix} \begin{pmatrix} -ie^{(-\gamma-\alpha)\frac{\pi}{4}i} & 0 \\ 0 & -ie^{(\gamma-\alpha)\frac{\pi}{4}i} \end{pmatrix} \tag{19}$$

$$M(t_{-1,0},0)=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{20}$$

Matrices $M(t,\infty)$ can be expressed in terms of Stokes coefficients, for every $t\in T$, so equality (22) of the next proposition provides relations that connect Stokes and connection coefficients.

Proposition 24. *Let $t,t'\in T$.*

$$\begin{array}{l} M(t't,0)=M(t',0)\,\tilde{t'}M(t,0) \\ M(t't,\infty)=M(t',\infty)\,\tilde{t'}M(t,\infty) \end{array} \tag{21}$$

$$\mathcal{L}M(t,0)=M(t,\infty)\tilde{t}(\mathcal{L}) \tag{22}$$

Proof : By applying t' to the relations $(ty_0,ty_{-1})=(y_0,y_{-1})M(t,\infty)$ and $(tN,tM)=(N,M)M(t,0)$ and using item 2 of the proposition 4 we obtain the first two relations of the proposition. In the same way with t and the connection relation $(N,M)=(y_0,y_{-1})\mathcal{L}$ we obtain (22).

The next proposition gives the fundamental relation of section 4.2.

Proposition 25. *For $(\alpha,\beta,\gamma,\delta,k)\in(\mathbb{C}-\mathbb{Z})\times\mathbb{C}^3\times\mathbb{Z}$ we have*

$$\det(\mathcal{L}(\alpha,\beta,\gamma,\delta)=-\frac{\alpha}{2} \tag{23}$$

Proof : From the connection relation we have

$$\begin{pmatrix} N & M \\ N' & M' \end{pmatrix}=\begin{pmatrix} y_0 & y_{-1} \\ y_0' & y_{-1}' \end{pmatrix}\mathcal{L}$$

The determinants are equal. Then relations (16) and (17) provide the result.

The following proposition gives a representation of each Stokes and connection coefficient in terms of l_{11}.

Proposition 26.

1. $l_{12}=\tilde{t}_{-1,0}l_{11}$
2. $l_{21}=ie^{(\gamma+\alpha)\frac{\pi}{4}i}\,\tilde{t}_{1,-1}l_{11}$
3. $l_{22}=ie^{(\gamma-\alpha)\frac{\pi}{4}i}\,\tilde{t}_{-1,-1}l_{11}$
4. $\frac{-\alpha}{2}e^{(-\alpha+\gamma)\frac{\pi}{2}i}c_0=\tilde{t}_{-1,0}l_{11}\,\tilde{t}_{1,2}l_{11}-e^{-\alpha\pi i}\,l_{11}\,\tilde{t}_{-1,2}l_{11}$

Proof :

Relation (22) for $t = t_{-1,0}$ and propositions 18 et 6 give

$$\mathcal{L}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \tilde{t}_{-1,0}\mathcal{L}$$

hence

$$\begin{aligned} l_{12} &= \tilde{t}_{-1,0} l_{11} \\ l_{21} &= \tilde{t}_{-1,0} l_{22} \end{aligned} \tag{24}$$

We have

$$M(t_{1,-1}, 0) = \begin{pmatrix} 1 & 0 \\ 0 & e^{\alpha\frac{\pi}{2}i} \end{pmatrix}$$

then, taking (18) into account, relation (22) for $t = t_{1,-1}$ can be written

$$\mathcal{L}\begin{pmatrix} 1 & 0 \\ 0 & e^{\alpha\frac{\pi}{2}i} \end{pmatrix} = \begin{pmatrix} -c_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} ie^{(\gamma+\alpha)\frac{\pi}{4}i} & 0 \\ 0 & ie^{(-\gamma+\alpha)\frac{\pi}{4}i} \end{pmatrix}\tilde{t}_{1,-1}\mathcal{L} \tag{25}$$

that is equivalent to the system

$$\left\{\begin{array}{llll} l_{11} = & -ie^{(\gamma+\alpha)\frac{\pi}{4}i}\, c_0 & \tilde{t}_{1,-1}l_{11} & +ie^{(-\gamma+\alpha)\frac{\pi}{4}i} \quad \tilde{t}_{1,-1}l_{21} \\ l_{12} = & -ie^{(\gamma-\alpha)\frac{\pi}{4}i} c_0 & \tilde{t}_{1,-1}l_{12} & +ie^{(-\gamma-\alpha)\frac{\pi}{4}i} \quad \tilde{t}_{1,-1}l_{22} \\ l_{21} = & ie^{(\gamma+\alpha)\frac{\pi}{4}i} & \tilde{t}_{1,-1}l_{11} & \\ l_{22} = & ie^{(\gamma-\alpha)\frac{\pi}{4}i} & \tilde{t}_{1,-1}l_{12} & \end{array}\right. \tag{26}$$

From (23) we deduce that $\det(\tilde{t}_{1,-1}\mathcal{L}) = -\frac{\alpha}{2}$

Then the system of the first two equations of the above system is equivalent to the following system

$$\left\{\begin{array}{lllll} -\frac{\alpha}{2}c_0 = & -ie^{(-\gamma+\alpha)\frac{\pi}{4}i} & l_{12}\ \tilde{t}_{1,-1}l_{21} & +ie^{(-\gamma-\alpha)\frac{\pi}{4}i} & l_{11}\ \tilde{t}_{1,-1}l_{22} \\ -\frac{\alpha}{2} = & -ie^{(\gamma+\alpha)\frac{\pi}{4}i} & l_{12}\ \tilde{t}_{1,-1}l_{11} & +ie^{(\gamma-\alpha)\frac{\pi}{4}i} & l_{11}\ \tilde{t}_{1,-1}l_{12} \end{array}\right. \tag{27}$$

Now, taking the two last equations of system (26) into account, we see that the second equation of the above system is nothing else but relation (23). Hence, according to (23), system (26) is equivalent to

$$\left\{\begin{array}{ll} -\frac{\alpha}{2}c_0 = & -ie^{(-\gamma+\alpha)\frac{\pi}{4}i}\, l_{12}\, \tilde{t}_{1,-1}l_{21} + ie^{(-\gamma-\alpha)\frac{\pi}{4}i}\, l_{11}\, \tilde{t}_{1,-1}l_{22} \\ l_{21} = & ie^{(\gamma+\alpha)\frac{\pi}{4}i}\, \tilde{t}_{1,-1}l_{11} \\ l_{22} = & ie^{(\gamma-\alpha)\frac{\pi}{4}i}\, \tilde{t}_{1,-1}l_{12} \end{array}\right. \tag{28}$$

Using relations (24) concludes the proof.

4.2 A fundamental relation for l_{11}

Relation (23) can be written

$$-\frac{\alpha}{2} = ie^{(\gamma-\alpha)\frac{\pi}{4}i}\, l_{11}\, \tilde{t}_{-1,-1}l_{11} - ie^{(\gamma+\alpha)\frac{\pi}{4}i}\, \tilde{t}_{-1,0}l_{11}\, \tilde{t}_{1,-1}l_{11} \tag{29}$$

The following proposition shows that this relation is sufficient to construct a family of possible Stokes and connection matrices.

Let us denote by $\mathcal{M}_{2\times 2}(\tilde{\Lambda})$ the set of (2×2) matrices with coefficients in $\tilde{\Lambda}$. Relation $t_{\epsilon',h'}t_{\epsilon,h} = e^{ih'\frac{\pi}{2}\frac{\epsilon-1}{2}\epsilon'\alpha}t_{\epsilon'\epsilon,h'+h}$ leads us to introduce the set $G = \{lt,\ (l,t)\in \tilde{\Lambda}\times T\}$.

Proposition 27. *Let λ be a function from $(\mathbb{C}-\mathbb{Z})\times\mathbb{C}^3$ into $\mathbb{C}$. Let γ_0 be the function from $(\mathbb{C}-\mathbb{Z})\times\mathbb{C}^3$ into $\mathbb{C}$ defined by*

$$\frac{-\alpha}{2}\gamma_0 = e^{(\alpha-\gamma)\frac{\pi}{2}i}\,\tilde{t}_{-1,0}\lambda\,\tilde{t}_{1,2}\lambda\ - e^{(-\alpha-\gamma)\frac{\pi}{2}i}\,\lambda\,\tilde{t}_{-1,2}\lambda \tag{30}$$

Let $\pounds$ be the matrix defined by

$$\pounds = \begin{pmatrix} \lambda & \tilde{t}_{-1,0}\lambda \\ ie^{(\gamma+\alpha)\frac{\pi}{4}i}\tilde{t}_{1,-1}\lambda & ie^{(\gamma-\alpha)\frac{\pi}{4}i}\tilde{t}_{-1,-1}\lambda \end{pmatrix} \tag{31}$$

If the relation

$$-\frac{\alpha}{2} = ie^{(\gamma-\alpha)\frac{\pi}{4}i}\,\lambda\,\tilde{t}_{-1,-1}\lambda\ - ie^{(\gamma+\alpha)\frac{\pi}{4}i}\,\tilde{t}_{-1,0}\lambda\,\tilde{t}_{1,-1}\lambda \tag{32}$$

is satisfied then there exists a unique family of matrices of $\mathcal{M}_{2\times 2}(\tilde{\Lambda})$, denoted by $(M(t,\infty,\lambda))_{t\in G}$, which satisfies the following relations

1.
$$\begin{aligned} &M(t_{1,-1},\infty,\lambda) = \begin{pmatrix} -\gamma_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} ie^{(\gamma+\alpha)\frac{\pi}{4}i} & 0 \\ 0 & ie^{(-\gamma+\alpha)\frac{\pi}{4}i} \end{pmatrix} \\ &M(t_{-1,0},\infty,\lambda) = I \\ &M(t_{1,0},\infty,\lambda) = I \end{aligned} \tag{33}$$

2. *For any $(l,t)\in\tilde{\Lambda}\times T$,*
$$M(lt,\infty,\lambda) = lM(t,\infty,\lambda) \tag{34}$$

3. *For any $t,t'\in T$*
$$M(t't,\infty,\lambda) = M(t',\infty,\lambda)\,\tilde{t}'M(t,\infty,\lambda) \tag{35}$$

Moreover for all $t\in T$ we have

$$\pounds M(t,0) = M(t,\lambda,\infty)\tilde{t}\pounds \tag{36}$$

Proof :

1. Unicity of the family $(M(t,\infty,\lambda))_{t\in G}$:

Assume the existence of a family $(M(t,\infty,\lambda))_{t\in G}$ described by the proposition. We have

$$I = M(t_{1,0},\infty,\lambda) = M(t_{1,1-1},\infty,\lambda) = M(t_{1,1},\infty,\lambda)\tilde{t}_{1,1}M(t_{1,-1},\infty,\lambda)$$

hence

$$M(t_{1,1},\infty,\lambda) = (\tilde{t}_{1,1}M(t_{1,-1},\infty,\lambda))^{-1} = \tilde{t}_{1,1}M(t_{1,-1},\infty,\lambda)^{-1} \tag{37}$$

that is equivalent, according to (33), to

$$M(t_{1,1},\infty,\lambda) = \begin{pmatrix} 0 & 1 \\ 1 & \gamma_{-1} \end{pmatrix} \begin{pmatrix} -ie^{(-\gamma-\alpha)\frac{\pi}{4}i} & 0 \\ 0 & -ie^{(\gamma-\alpha)\frac{\pi}{4}i} \end{pmatrix} \tag{38}$$

in which $\gamma_{-1} = e^{-\gamma\frac{\pi}{2}i}\,\tilde{t}_{1,1}\gamma$.

We remark that, for $j \in \{0,-1\}$,

$$\gamma_j = \tilde{t}_{-1,0}\gamma_j \tag{39}$$

Let $t \in T$.

If $t = t_{1,h}$ with $h \in \mathbb{N}$, then by induction we have

$$\begin{aligned} M(t_{1,h},\infty,\lambda) &= M(t_{1,1},\infty,\lambda)\tilde{t}_{1,1}M(t_{1,h-1},\infty,\lambda) = ... \\ &= M(t_{1,1},\infty,\lambda)\tilde{t}_{1,1}M(t_{1,1},\infty,\lambda)...\tilde{t}_{1,1}^{h-1}M(t_{1,1},\infty,\lambda) \end{aligned} \tag{40}$$

If $t = t_{1,-h}$ with $h \in \mathbb{N}$,

$$M(t_{1,-h},\infty,\lambda) = M(t_{1,-1},\infty,\lambda)\tilde{t}_{1,-1}M(t_{1,-1},\infty,\lambda)...\tilde{t}_{1,-1}^{h-1}M(t_{1,-1},\infty,\lambda). \tag{41}$$

If $t = t_{-1,h}$ with $h \in \mathbb{Z}$, we have $M(t_{-1,h},\infty,\lambda) = M(t_{-1,0},\infty,\lambda)\tilde{t}_{-1,0}M(t_{1,h},\infty,\lambda)$ then

$$M(t_{-1,h},\infty,\lambda) = \tilde{t}_{-1,0}M(t_{1,h},\infty,\lambda). \tag{42}$$

Finally, if $(l,t) \in \tilde{\Lambda} \times T$, we have $M(lt,\infty,\lambda) = lM(t,\infty,\lambda)$.

From the above we deduce the unicity of the family $(M(t,\infty,\lambda))_{t\in G}$.

2. Existence of $(M(t,\infty,\lambda))_{t\in G}$ **:**

We define $M(t,\infty,\lambda)$ as following,

$M(t_{1,-1},\infty,\lambda)$, $M(t_{1,0},\infty,\lambda)$, $M(t_{-1,0},\infty,\lambda)$ by (33),

$M(t_{1,1},\infty,\lambda)$ by (38),

$M(t_{1,h},\infty,\lambda)$ by (40) et (41),

$M(t_{-1,h},\infty,\lambda)$ by (42),

$M(lt,\infty,\lambda) = lM(t,\infty,\lambda)$

We have to prove that this family satisfies relations (35). In the sequel, we shall often write $M(t_{\epsilon,h})$ instead of $M(t_{\epsilon,h},\infty,\lambda)$. Let us first prove that for each $h \in \mathbb{Z}$

$$\tilde{t}_{-1,0}M(t_{1,h}) = e^{h\frac{\pi}{2}\alpha i}M(t_{1,h}) \tag{43}$$

By using definitions (33) and (38) of $M(t_{1,1})$ and $M(t_{1,-1})$ and relation (39) we obtain (43) for $h = 1$ and $h = -1$. By induction we conclude.

From relation (43) and definition of $M(t_{-1,h})$ we obtain

$$M(t_{-1,h}) = e^{h\frac{\pi}{2}\alpha i}M(t_{1,h}) \tag{44}$$

We are now going to prove relation (35) when (t',t) becomes in turn equal to $(t_{1,h'},\ t_{1,h})$, $(t_{-1,h'},\ t_{1,h})$, $(t_{1,h'},\ t_{-1,h})$, $(t_{-1,h'},\ t_{-1,h})$.

- Let us prove relation

$$M(t_{1,h'}t_{1,h}) = M(t_{1,h'})\tilde{t}_{1,h'}M(t_{1,h}) \tag{45}$$

From their definitions, $M(t_{1,1})$ and $M(t_{1,-1})$ verify (37). As a result we have

$$M(t_{1,1})\tilde{t}_{1,1}M(t_{1,-1}) = M(t_{1,-1})\tilde{t}_{1,-1}M(t_{1,1}) = I \tag{46}$$

By induction we can prove relation (45) for the two cases $hh' > 0$ and $hh' < 0$.

- Let us prove relation $M(t_{-1,h'}t_{1,h}) = M(t_{-1,h'})\tilde{t}_{-1,h'}M(t_{1,h})$.

Definition (42) gives $M(t_{-1,h'}t_{1,h}) = M(t_{-1,h'+h}) = \tilde{t}_{-1,0}M(t_{1,h'+h})$.

We also have
$M(t_{-1,h'})\tilde{t}_{-1,h'}M(t_{1,h}) = \tilde{t}_{-1,0}M(t_{1,h'})\tilde{t}_{-1,0}\tilde{t}_{1,h'}M(t_{1,h}) =$

$$\tilde{t}_{-1,0}\{M(t_{1,h'})\tilde{t}_{1,h'}M(t_{1,h})\} = \tilde{t}_{-1,0}M(t_{-1,h'+h})$$

- Let us prove relation $M(t_{1,h'}t_{-1,h}) = M(t_{1,h'})\tilde{t}_{1,h'}M(t_{-1,h})$.

Using (34) and (44) we obtain
$M(t_{1,h'}t_{-1,h}) = M(e^{-h'\frac{\pi}{2}\alpha i}t_{-1,h'+h}) = e^{-h'\frac{\pi}{2}\alpha i}M(t_{-1,h'+h}) =$

$$e^{-h'\frac{\pi}{2}\alpha i}e^{(h'+h)\frac{\pi}{2}\alpha i}M(t_{1,h'+h}) = e^{h\frac{\pi}{2}\alpha i}M(t_{1,h'+h}).$$

Using (44) and (45) we have:
$M(t_{1,h'})\tilde{t}_{1,h'}M(t_{-1,h}) = M(t_{1,h'})\tilde{t}_{1,h'}e^{h\frac{\pi}{2}\alpha i}M(t_{1,h}) =$

$$e^{h\frac{\pi}{2}\alpha i}M(t_{1,h'})\tilde{t}_{1,h'}M(t_{1,h}) = e^{h\frac{\pi}{2}\alpha i}M(t_{1,h'+h}).$$

- Let us prove the relation $M(t_{-1,h'}t_{-1,h}) = M(t_{-1,h'})\tilde{t}_{-1,h'}M(t_{-1,h})$.

From (34) we have: $M(t_{-1,h'}t_{-1,h}) = M(e^{h'\frac{\pi}{2}\alpha i}t_{1,h'+h}) = e^{h'\frac{\pi}{2}\alpha i}M(t_{1,h'+h})$.

Using (44) and (45) we obtain:
$M(t_{-1,h'})\tilde{t}_{-1,h'}M(t_{-1,h}) = M(t_{-1,h'})\tilde{t}_{1,h'}\tilde{t}_{-1,0}\tilde{t}_{-1,0}M(t_{1,h}) =$

$$e^{h'\frac{\pi}{2}\alpha i}M(t_{1,h'})\tilde{t}_{1,h'}M(t_{1,h}) = e^{h'\frac{\pi}{2}\alpha i}M(t_{1,h'+h}).$$

3. The family $(M(t,\infty,\lambda))_{t\in G}$ satisfies relations (36) :

We need the following lemma.

Lemma 28. *Let $(M(t,\infty,\lambda))_{t\in G}$ be the family defined above. Let $t,\ t' \in T$. If*

1. $\pounds M(t,0) = M(t,\infty,\lambda)\tilde{t}\pounds$,
2. $\pounds M(t',0) = M(t',\infty,\lambda)\tilde{t}'\pounds$,

then we have
$\pounds M(t't,0) = M(t't,\infty,\lambda)\tilde{t}'\tilde{t}\pounds$

Proof : By using the assumptions of the lemma, the first relation of (21) and relations (35) we obtain:

$$\pounds M(t't,0) = \pounds M(t',0)\ \tilde{t}' M(t,0) = M(t',\infty,\lambda)\ \tilde{t}'\pounds\ \tilde{t}' M(t,0) =$$
$$M(t',\infty,\lambda)\ \tilde{t}'\{\pounds M(t,0)\} = M(t',\infty,\lambda)\ \tilde{t}'\{M(t,\infty,\lambda)\tilde{t}\pounds\} =$$
$$M(t',\infty,\lambda)\ \tilde{t}' M(t,\infty,\lambda)\ \tilde{t}'\tilde{t}\pounds = M(t't,\infty,\lambda)\ \tilde{t}'\tilde{t}\pounds.$$

- For $t = t_{-1,0}$ relation (36) can be written $\pounds \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \tilde{t}_{-1,0}\pounds$. We easily verify that it is true.
- For $t = t_{1,-1}$ we prove relation (36) by putting $(\gamma_0, \pounds)$ instead of $(c_0, \mathcal{L})$ in the proof of proposition 26 and by using assumption (23) which means that $\det(\pounds) = -\frac{\alpha}{2}$.
- For $t = t_{1,1}$, we apply $t_{1,1}$ to $\pounds M(t_{1,-1},0) = M(t_{1,-1})\pounds$ and use (37) to obtain (36).

Relations (36) are true for $t_{1,1}$, $t_{1,-1}$ and $t_{-1,0}$. By using (2) and lemma 28 we prove that, for all $t \in T$, (36) is true.

4.3 Some relations deduced from the fundamental relation

Equality (22), with a particular $t \in T$, provides some new relations satisfied by the Stokes and connection coefficients. We first obtain relation (47) which is, taking (14) into account, a functional relation for c_0.

Proposition 29. *Let $(\alpha,\beta,\gamma,\delta) \in (\mathbb{C} - \mathbb{Z}) \times \mathbb{C}^3$. The Stokes coefficients c_k satisfy*

$$c_0c_1 + c_0c_3 + c_2c_3 + c_0c_1c_2c_3 + e^{i\pi 2\gamma}c_1c_2 = 2e^{i\pi\gamma}(\cos\pi\alpha - \cos\pi\gamma) \qquad (47)$$

Proof : By using propositions 18 and 6 and the Stokes matrices we obtain

$$M(t_{1,-4},0) = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\alpha\pi i} \end{pmatrix} \qquad (48)$$

$$M(t_{1,-4},\infty) = \mathcal{M}_0^{-1}\mathcal{M}_1^{-1}\mathcal{M}_2^{-1}\mathcal{M}_3^{-1} \begin{pmatrix} e^{(-\gamma+\alpha)\pi i} & 0 \\ 0 & e^{(\gamma+\alpha)\pi i} \end{pmatrix} \qquad (49)$$

Then, by (22) for $t = t_{1,-4}$, we have

$$\mathcal{L}\begin{pmatrix} 1 & 0 \\ 0 & e^{2\alpha\pi i} \end{pmatrix} = \mathcal{M}_0^{-1}\mathcal{M}_1^{-1}\mathcal{M}_2^{-1}\mathcal{M}_3^{-1} \begin{pmatrix} e^{(-\gamma+\alpha)\pi i} & 0 \\ 0 & e^{(\gamma+\alpha)\pi i} \end{pmatrix}\mathcal{L} \qquad (50)$$

The matrices $M(t_{1,-4},0)$ and $M(t_{1,-4},\infty)$ have the same trace. It implies the given relation.

With the next proposition we try to obtain the connection coefficients in terms of c_0.

Proposition 30. *Let* $(\alpha, \beta, \gamma, \delta) \in (\mathbb{C} - \mathbb{Z}) \times \mathbb{C}^3$. *We have*

$$l_{11}\,\tilde{t}_{-1,0} l_{11} = \frac{\alpha e^{(-\alpha+\gamma)\pi i}(c_0 + c_2 + c_0c_1c_2)}{2(-1 + e^{-2\alpha\pi i})} \tag{51}$$

Proof :
According to proposition 26 we can write

$$\mathcal{L} = \begin{pmatrix} l_{11} & \tilde{t}_{-1,1} l_{11} \\ e^{(\alpha+\gamma)\frac{\pi}{4}i}\tilde{t}_{1,-1} l_{11} & e^{(-\alpha+\gamma)\frac{\pi}{4}i}\tilde{t}_{-1,-1} l_{11} \end{pmatrix} \tag{52}$$

By using (47), we prove that relation (50) is equivalent to the following system

$$\begin{cases} (c_1c_2 + 1 - e^{(\alpha-\gamma)\pi i}) l_{11} + i(c_0 + c_2 + c_0c_1c_2) e^{(\alpha+\gamma)\frac{\pi}{4}i}\tilde{t}_{1,-1} l_{11} = 0 \\ (c_1c_2 + 1 - e^{(-\alpha-\gamma)\pi i})\tilde{t}_{-1,0} l_{11} + i(c_0 + c_2 + c_0c_1c_2) e^{(-\alpha+\gamma)\frac{\pi}{4}i}\tilde{t}_{-1,-1} l_{11} = 0 \\ (c_1 + c_3 + c_1c_2c_3) e^{-2\gamma\pi i} l_{11} - ie^{(\alpha+\gamma)\frac{\pi}{4}i}(-e^{i\pi(-\alpha-\gamma)} + 1 + c_1c_2)\tilde{t}_{1,-1} l_{11} = 0 \\ (c_1 + c_3 + c_1c_2c_3) e^{-2\gamma\pi i}\tilde{t}_{-1,0} l_{11} - ie^{(-\alpha+\gamma)\frac{\pi}{4}i}(-e^{i\pi(\alpha-\gamma)} + 1 + c_1c_2)\tilde{t}_{-1,-1} l_{11} = 0 \end{cases} \tag{53}$$

- If $c_0 + c_2 + c_0c_1c_2 \neq 0$ then from the equations 1 and 2 of the system (53) we get

$$\tilde{t}_{1,-1} l_{11} = \frac{(c_1c_2 + 1 - e^{(\alpha-\gamma)\pi i}) l_{11}}{i(c_0 + c_2 + c_0c_1c_2) e^{(\alpha+\gamma)\frac{\pi}{4}i}}$$

  and

$$\tilde{t}_{-1,-1} l_{11} = \frac{(c_1c_2 + 1 - e^{(-\alpha-\gamma)\pi i})\tilde{t}_{-1,1} l_{11}}{i(c_0 + c_2 + c_0c_1c_2) e^{(-\alpha+\gamma)\frac{\pi}{4}i}}$$

  By using these identities in the fundamental relation (29) we obtain (51).

- If $c_0 + c_2 + c_0c_1c_2 = 0$ then from the first equality of the system we have the two following cases

  If $l_{11} = 0$ then we have (51).

  If $c_1c_2 + 1 - e^{(\alpha-\gamma)\pi i} = 0$ then $c_1c_2 + 1 - e^{(-\alpha-\gamma)\pi i} = 2isin(\alpha\pi)e^{-\gamma\pi i} \neq 0$ $(\alpha \notin \mathbb{Z})$. Then equation 2 of system (53) implies $\tilde{t}_{-1,0} l_{11} = 0$ and (51).

Proposition 31. *Let* $(\alpha, \beta, \gamma, \delta) \in (\mathbb{C} - \mathbb{Z}) \times \mathbb{C}^3$. *We have*

$$l_{11}(\alpha, \beta, \gamma, \delta) = 0 \Longleftrightarrow \begin{cases} c_2 = -c_0 e^{i\pi(-\alpha-\gamma)} \\ c_1c_2 = -1 + e^{i\pi(-\alpha-\gamma)} \end{cases}$$

Proof :
By (51) and (29), $l_{11}(\alpha, \beta, \gamma, \delta) = 0$ is equivalent to

$$\begin{cases} c_0 + c_2 + c_0c_1c_2 = 0 \\ l_{11}(-\alpha, \beta, \gamma, \delta) \neq 0 \end{cases}$$

which is, by the second equation of system (53), equivalent to

$$\begin{cases} c_0 + c_2 + c_0 c_1 c_2 = 0 \\ c_1 c_2 + 1 - e^{i\pi(-\alpha-\gamma)} = 0 \end{cases}$$

which is equivalent to

$$\begin{cases} c_2 = -c_0 e^{i\pi(-\alpha-\gamma)} \\ c_1 c_2 = -1 + e^{i\pi(-\alpha-\gamma)} \end{cases}$$

Comparison with the coefficient K_2 of P. Maroni

P. Maroni defines, in [5] (pages 214, 227) $K_1(\alpha,\beta,\gamma,\delta)$, $K_2(\alpha,\beta,\gamma,\delta)$, $M_1^+(\alpha,\beta,\gamma,\delta)$, $M_2^+(\alpha,\beta,\gamma,\delta)$ by $N = K_1 B^+ + K_2 H^+$ and $B^+ = M_1^+ N + M_2^+ M$.

We use proposition 23 to easily prove $M_+^1 = -\frac{2}{\alpha}\tilde{t}_{-1,0} l_{21}$ and $M_+^2 = \frac{2}{\alpha} l_{21}$. Thus, we can complete lemma 4.4.1 of [5], page 227, by the following identities

$$K_2 = \frac{\alpha}{2}\tilde{t}_{-1,0} M_+^1 = l_{21}$$

5 Conclusion

The integral representations of the k-sums at ∞ of a formal power series are, usually, only defined in the vicinity of ∞. For the solutions of the BHE, the integral representations obtained are defined in sectors of infinite radius. We have also shown the same phenomenon for the double confluent Heun equations (*c.f* [8]) and for the triconfluent Heun equations.

Except for the $BHE(\alpha, 0, \gamma, 0)$, which are the Kummer equations, we cannot express the Stokes and connection coefficients of the BHE in terms of any known functions. As these coefficients are analytic functions in the parameters it would be useful to know if l_{11} is the only analytic function in the parameters which satisfies the fundamental relation (32).

6 Appendix

6.1 Differential equation satisfied by $\sum_{n=0}^{\infty} \frac{a_{2n}}{\Gamma(n+1)} t^n$

Coefficient of $z(x)$:

$$-\frac{1}{32}(2+\alpha-\gamma)(4+\alpha-\gamma)(-4+\alpha+\gamma)(-2+\alpha+\gamma)(-7\beta+\beta\gamma+\delta)$$

Coefficient of $z'(x)$:

$$\frac{1}{8}(1096\beta-46\alpha^2\beta-105\beta^3-720\beta\gamma+10\alpha^2\beta\gamma+71\beta^3\gamma+150\beta\gamma^2-15\beta^3\gamma^2-10\beta\gamma^3+\beta^3\gamma^3-280\delta+10\alpha^2\delta+71\beta^2\delta+104\gamma\delta-30\beta^2\gamma\delta-10\gamma^2\delta+3\beta^2\gamma^2\delta-15\beta\delta^2+3\beta\gamma\delta^2+\delta^3)+\frac{1}{8}(6224\beta-240\alpha^2\beta+\alpha^4\beta-4088\beta\gamma+76\alpha^2\beta\gamma+1024\beta\gamma^2-6\alpha^2\beta\gamma^2-116\beta\gamma^3+5\beta\gamma^4-560\delta+20\alpha^2\delta+312\gamma\delta-4\alpha^2\gamma\delta-60\gamma^2\delta+4\gamma^3\delta)x$$

Coefficient of $z''(x)$:

$$-24(-3\beta+\beta\gamma+\delta)+\frac{1}{2}(2884\beta-28\alpha^2\beta-147\beta^3-900\beta\gamma+2\alpha^2\beta\gamma+42\beta^3\gamma+84\beta\gamma^2-3\beta^3\gamma^2-2\beta\gamma^3-308\delta+2\alpha^2\delta+42\beta^2\delta+56\gamma\delta-6\beta^2\gamma\delta-2\gamma^2\delta-3\beta\delta^2)x+(2590\beta-25\alpha^2\beta-940\beta\gamma+3\alpha^2\beta\gamma+117\beta\gamma^2-5\beta\gamma^3-154\delta+\alpha^2\delta+42\gamma\delta-3\gamma^2\delta)x^2$$

Coefficient of $z^{(3)}(x)$:

$$-12(-29\beta+3\beta\gamma+3\delta)x+2(998\beta-2\alpha^2\beta-27\beta^3-162\beta\gamma+3\beta^3\gamma+6\beta\gamma^2-54\delta+3\beta^2\delta+4\gamma\delta)x^2-4(-496\beta+\alpha^2\beta+98\beta\gamma-5\beta\gamma^2+18\delta-2\gamma\delta)x^3$$

Coefficient of $z^{(4)}(x)$:

$$-8(-29\beta+\beta\gamma+\delta)x^2-8(-59\beta+5\beta\gamma+\delta)x^3-8(-88\beta+\beta^3+6\beta\gamma+2\delta)x^4$$

Coefficient of $z^{(5)}(x)$:

$$32\ x^3(1+x)^2$$

6.2 Differential equation satisfied by $\sum_{n=0}^{\infty}\frac{a_{2n+1}}{\Gamma(n+1)}t^n$

Coefficient of $z(x)$:

$$-\frac{1}{32}(4+\alpha-\gamma)(6+\alpha-\gamma)(-6+\alpha+\gamma)(-4+\alpha+\gamma)(-9\beta+\beta\gamma+\delta)$$

Coefficient of $z'(x)$:

$$\frac{1}{8}(4504\beta-94\alpha^2\beta-315\beta^3-2016\beta\gamma+14\alpha^2\beta\gamma+143\beta^3\gamma+294\beta\gamma^2-21\beta^3\gamma^2-14\beta\gamma^3+\beta^3\gamma^3-728\delta+14\alpha^2\delta+143\beta^2\delta+200\gamma\delta-42\beta^2\gamma\delta-14\gamma^2\delta+3\beta^2\gamma^2\delta-21\beta\delta^2+3\beta\gamma\delta^2+\delta^3)+\frac{1}{8}(19504\beta-416\alpha^2\beta+\alpha^4\beta-9736\beta\gamma+100\alpha^2\beta\gamma+1840\beta\gamma^2-6\alpha^2\beta\gamma^2-156\beta\gamma^3+5\beta\gamma^4-1456\delta+28\alpha^2\delta+600\gamma\delta-4\alpha^2\gamma\delta-84\gamma^2\delta+4\gamma^3\delta)x$$

Coefficient of $z''(x)$:

$$-40(-5\beta+\beta\gamma+\delta)+\frac{1}{2}(6012\beta-36\alpha^2\beta-243\beta^3-1476\beta\gamma+2\alpha^2\beta\gamma+54\beta^3\gamma+108\beta\gamma^2-3\beta^3\gamma^2-2\beta\gamma^3-500\delta+2\alpha^2\delta+54\beta^2\delta+72\gamma\delta-6\beta^2\gamma\delta-2\gamma^2\delta-3\beta\delta^2)x+(4978\beta-31\alpha^2\beta-1468\beta\gamma+3\alpha^2\beta\gamma+147\beta\gamma^2-5\beta\gamma^3-250\delta+\alpha^2\delta+54\gamma\delta-3\gamma^2\delta)x^2$$

Coefficient of $z^{(3)}(x)$:

$$-4(-139\beta+11\beta\gamma+11\delta)x+2(1478\beta-2\alpha^2\beta-33\beta^3-198\beta\gamma+3\beta^3\gamma+6\beta\gamma^2-66\delta+3\beta^2\delta+4\gamma\delta)x^2-4(-712\beta+\alpha^2\beta+118\beta\gamma-5\beta\gamma^2+22\delta-2\gamma\delta)x^3$$

Coefficient of $z^{(4)}(x)$:

$$-8(-35\beta+\beta\gamma+\delta)x^2-8(-104\beta+\beta^3+6\beta\gamma+2\delta)x^3-8(-69\beta+5\beta\gamma+\delta)x^4$$

Coefficient of $z^{(5)}(x)$:

$$32\beta x^3(1+x)^2$$

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