On the selection of basic orthogonal sequences in non-archimedean metrizable locally convex spaces

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Abstract

Our main result (Theorem 1) follows that any infinite-dimensional subspace F of a non-archimedean metrizable locally convex space E with an orthogonal basis (e_n) contains a basic orthogonal sequence equivalent to a block basic orthogonal sequence relative to (e_n) (Proposition 2). Hence any infinite-dimensional non-archimedean metrizable locally convex space F possesses a basic orthogonal sequence equivalent to a block basic orthogonal sequence relative to an orthogonal basis in $c_0^{\mathbb{N}}$ (Corollary 3).

Introduction

In this paper all linear spaces are over a non-archimedean non-trivially valued field $\mathbb{K}$ which is complete under the metric induced by the valuation $|\cdot| : \mathbb{K} \to [0, \infty)$. For fundamentals of locally convex Hausdorff spaces (lcs) and normed spaces we refer to [2], [4] and [3]. Orthogonal bases and basic orthogonal sequences in locally convex spaces are studied in [1], [6] and [8].

Any infinite-dimensional Banach space of countable type is isomorphic to the Banach space c_0 of all sequences in $\mathbb{K}$ converging to zero (with the sup-norm) ([3], Theorem 3.16), so it has an orthogonal basis.

There exist infinite-dimensional Fréchet spaces of countable type without a Schauder basis (see [7]). Nevertheless, any infinite-dimensional metrizable lcs E of finite

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type has an orthogonal basis ([1], Theorem 3.5). Moreover, any infinite-dimensional metrizable lcs E possesses a basic orthogonal sequence ([6], Theorem 2). It is also known that any bounded non-compactoid subset in a lcs E contains a basic orthogonal sequence in E ([1], Theorem 2.2).

In this note we are interested in the selection of basic orthogonal sequences in metrizable locally convex spaces.

Using the stability theorem for basic orthogonal sequences in metrizable lcs ([8], Corollary 2), we show our main result: Let E be a metrizable lcs with an orthogonal basis (e_n) and let (f_n) be the sequence of coefficient functionals associated with the basis (e_n). If $(y_n) \subset E, y_n \not\to 0$ and $\lim_n f_j(y_n) = 0$ for any $j \in \mathbb{N}$, then (y_n) has a subsequence (y_{i_n}) which is a basic orthogonal sequence equivalent to a block basic orthogonal sequence relative to (e_n) (Theorem 1). It follows that any infinite-dimensional subspace F of a metrizable lcs E with an orthogonal basis (e_n) contains a basic orthogonal sequence equivalent to a block basic orthogonal sequence relative to (e_n) (Proposition 2). Thus any infinite-dimensional metrizable lcs F has a basic orthogonal sequence equivalent to a block basic orthogonal sequence relative to an orthogonal basis in $c_0^{\mathbb{N}}$ (Corollary 3).

Preliminaries

Let E, F be locally convex spaces. A map $T : E \to F$ is called an *isomorphism* if T is linear, one-to-one, surjective and the maps T, T^{-1} are continuous.

A sequence (x_n) in a lcs E is *equivalent* to a sequence (y_n) in a lcs F if there exists an isomorphism T between the linear spans of (x_n) and (y_n), such that $Tx_n = y_n$ for all $n \in \mathbb{N}$.

A sequence (x_n) in a lcs E is a *Schauder basis* in E if each $x \in E$ can be written uniquely as $x = \sum_{n=1}^{\infty} \alpha_n x_n$ with $(\alpha_n) \subset \mathbb{K}$ and the coefficient functionals $f_n : E \to \mathbb{K}, x \to \alpha_n \, (n \in \mathbb{N})$ are continuous.

By a *seminorm* on a linear space E we mean a function $p : E \to [0, \infty)$ such that $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{K}, x \in E$ and $p(x + y) \le \max\{p(x), p(y)\}$ for all $x, y \in E$. A seminorm p on E is a *norm* if $\ker p := \{x \in E : p(x) = 0\} = \{0\}$.

Let p be a seminorm on a linear space E. A sequence $(x_n) \subset E$ is 1-*orthogonal with respect to* p if $p(\sum_{i=1}^{n} \alpha_i x_i) = \max_{1 \le i \le n} p(\alpha_i x_i)$ for all $n \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$.

Let E be a metrizable lcs.

The set of all continuous seminorms on E is denoted by $\mathcal{P}(E)$. A non-decreasing sequence $(p_k) \subset \mathcal{P}(E)$ is a *base* in $\mathcal{P}(E)$ if for every $p \in \mathcal{P}(E)$ there exists $k \in \mathbb{N}$ with $p \le p_k$.

A sequence $(x_n) \subset E$ is 1-*orthogonal with respect to* $(p_k) \subset \mathcal{P}(E)$ if (x_n) is 1-orthogonal with respect to p_k for every $k \in \mathbb{N}$. A sequence $(x_n) \subset (E \setminus \{0\})$ is a *basic orthogonal sequence* in E if it is 1-orthogonal with respect to some base (p_k) in $\mathcal{P}(E)$ (this concept coincides with the one given in [8] in the general context of lcs). A basic orthogonal sequence in a subspace F of E is a basic orthogonal sequence in E ([1], Remark 1.2). A linearly dense basic orthogonal sequence in E is an *orthogonal basis* in E. Any orthogonal basis in E is a Schauder basis ([1], Proposition 1.4) and any Schauder basis in a Fréchet space is an orthogonal basis ([1], Proposition 1.7).

Let (x_n) be a basic orthogonal sequence in E. Let $(k_n) \subset \mathbb{N}$ be an increasing sequence and let $(\alpha_n) \subset \mathbb{K}$ with $\max_{k_n \le i < k_{n+1}} |\alpha_i| > 0$ for any $n \in \mathbb{N}$. Put $y_n = \sum_{i=k_n}^{k_{n+1}-1} \alpha_i x_i, n \in \mathbb{N}$. The sequence (y_n) is said to be a *block basic orthogonal sequence relative to* (x_n).

E is *of finite type* if for any $p \in \mathcal{P}(E)$ the quotient space $(E/\ker p)$ is finite-dimensional.

E is *of countable type* if it contains a linearly dense countable subset (this notion agrees with the one of lcs of countable type given in [4]).

A *Fréchet space* is a metrizable complete lcs.

Let (x_n) be a sequence in a Fréchet space F. The series $\sum_{n=1}^{\infty} x_n$ is convergent in F if and only if $\lim x_n = 0$.

Results

We start with the following.

Theorem 1. *Let E be a metrizable lcs with an orthogonal basis (e_n) and let (f_n) be the sequence of coefficient functionals associated with the basis (e_n). If $(y_n) \subset E, y_n \not\to 0$ and $\lim_n f_j(y_n) = 0$ for any $j \in \mathbb{N}$, then (y_n) has a subsequence (y_{i_n}) which is a basic orthogonal sequence equivalent to a block basic orthogonal sequence relative to (e_n).*

Proof. The basis (e_n) is 1-orthogonal with respect to some base (p_k) in $\mathcal{P}(E)$. Without loss of generality we can assume that $\inf_n p_1(y_n) \ge 1$. By induction we can construct two increasing sequences $(d_n), (t_n) \subset \mathbb{N}$ with $d_1 = t_1 = 1$ such that for any $n \in \mathbb{N}$ we have

$$(*) \quad \max_{1 \le k \le n} \max_{j \in (\mathbb{N} \setminus \{t_n+1, \dots, t_{n+1}\})} p_k(f_j(y_{d_{n+1}})e_j) < 1.$$

Indeed, assume that for some $m \in \mathbb{N}$ we have already chosen $d_1, t_1, \dots, d_m, t_m \in \mathbb{N}$ with $1 = d_1 < \cdots < d_m, 1 = t_1 < \cdots < t_m$ such that for any $n \in \mathbb{N}$ with $n < m$ holds $(*)$. Since $\lim_n f_j(y_n) = 0$ for any $j \in \mathbb{N}$, there is $d_{m+1} \in \mathbb{N}$ with $d_{m+1} > d_m$ such that $\max_{1 \le k \le m} \max_{1 \le j \le t_m} p_k(f_j(y_{d_{m+1}})e_j) < 1$. Clearly $\lim_j f_j(y_{d_{m+1}})e_j = 0$. Hence there is $t_{m+1} \in \mathbb{N}$ with $t_{m+1} > t_m$ such that $\max_{1 \le k \le m} \max_{j > t_{m+1}} p_k(f_j(y_{d_{m+1}})e_j) < 1$.

Put $i_n = d_{n+1}$ and $x_n = \sum_{j=t_n+1}^{t_{n+1}} f_j(y_{i_n})e_j$ for $n \in \mathbb{N}$. Using $(*)$ and the inequality $\inf_n p_1(y_n) \ge 1$ we obtain $\max_{t_n+1 \le j \le t_{n+1}} |f_j(y_{i_n})| > 0, n \in \mathbb{N}$. Then (x_n) is a block basic orthogonal sequence relative to (e_n); of course, it is 1-orthogonal with respect to (p_k). Let $k, n \in \mathbb{N}$ with $k \le n$. By $(*)$ we get

$$p_k(x_n - y_{i_n}) = p_k(\sum_{j=1}^{t_n} f_j(y_{i_n})e_j + \sum_{j=t_{n+1}+1}^{\infty} f_j(y_{i_n})e_j) =$$

$$\max_{j \in (\mathbb{N} \setminus \{t_n+1, \dots, t_{n+1}\})} p_k(f_j(y_{i_n})e_j) < 1 \le p_k(y_{i_n}).$$

By the strong triangle inequality for p_k we have $p_k(x_n) = p_k(y_{i_n})$. Thus $p_k(x_n - y_{i_n}) < p_k(x_n)$ for all $k, n \in \mathbb{N}$ with $k \le n$. By [8], Corollary 2, (y_{i_n}) is a basic orthogonal sequence equivalent to (x_n). ∎

Using Theorem 1 we get

Proposition 2. *Let E be a metrizable lcs with an orthogonal basis (e_n). Then any infinite-dimensional subspace F of E contains a basic orthogonal sequence equivalent to a block basic orthogonal sequence relative to (e_n).*

Proof. Consider two cases.

Case 1: F is of finite type. Then F has an orthogonal basis (y_n) ([1], Theorem 3.5). The basis (y_n) is 1-orthogonal with respect to some base (q_k) in $\mathcal{P}(F)$. Since $\dim(F/\ker q_k) < \infty, k \in \mathbb{N}$, then the set $\{n \in \mathbb{N} : q_k(y_n) > 0\}$ is finite for all $k \in \mathbb{N}$. Hence for any $(\alpha_n) \subset \mathbb{K}$ the series $\sum_{n=1}^{\infty} \alpha_n y_n$ is convergent in the completion $\tilde{F}$ of F.

The basis (e_n) is 1-orthogonal with respect to some base (p_k) in $\mathcal{P}(E)$. Clearly, $\dim\ker p_k = \infty$ for any $k \in \mathbb{N}$; so for every $(k, r) \in \mathbb{N} \times \mathbb{N}$ there is $n \in \mathbb{N}$ with $n > r$ such that $e_n \in \ker p_k$. Thus there exist two increasing sequences $(d_n), (t_n) \subset \mathbb{N}$ such that $e_{d_n} \in (\ker p_{t_n} \setminus \ker p_{t_{n+1}}), n \in \mathbb{N}$. Then for any $(\alpha_n) \subset \mathbb{K}$ the series $\sum_{n=1}^{\infty} \alpha_n e_{d_n}$ is convergent in the completion $\tilde{H}$ of the closed linear span H of (e_{d_n}).

By the closed graph theorem ([2], Theorem 2.49) the linear maps $T : \mathbb{K}^{\mathbb{N}} \to \tilde{F}, (\alpha_n) \to \sum_{n=1}^{\infty} \alpha_n y_n$ and $S : \mathbb{K}^{\mathbb{N}} \to \tilde{H}, (\alpha_n) \to \sum_{n=1}^{\infty} \alpha_n e_{d_n}$ are isomorphisms. It follows that the basic orthogonal sequence (y_n) is equivalent to (e_{d_n}). Of course, (e_{d_n}) is a block basic orthogonal sequence relative to (e_n).

Case 2: F is not of finite type. Then $\dim(F/\ker p) = \infty$ for some $p \in \mathcal{P}(F)$. Let (f_n) be the sequence of coefficient functionals associated with the basis (e_n). Put $U = \{x \in F : p(x) < 1\}$ and $F_n = \bigcap_{j=1}^{n} \ker f_j \cap F, n \in \mathbb{N}$. Since $\dim(F/F_n) < \infty$, then $(F_n \setminus U) \neq \emptyset$ for any $n \in \mathbb{N}$. Let $y_n \in (F_n \setminus U), n \in \mathbb{N}$. Clearly, $(y_n) \subset F, y_n \not\to 0$ and $\lim_n f_j(y_n) = 0$ for any $j \in \mathbb{N}$. By Theorem 1, (y_n) contains a subsequence (y_{i_n}) which is a basic orthogonal sequence equivalent to a block basic orthogonal sequence relative to (e_n). ∎

Since any metrizable lcs E of countable type is isomorphic to a subspace of the Fréchet space $c_0^{\mathbb{N}}$ ([1], Remark 3.6) we get

Corollary 3. *Any infinite-dimensional metrizable lcs contains a basic orthogonal sequence equivalent to a block basic orthogonal sequence relative to an orthogonal basis in $c_0^{\mathbb{N}}$.*

It is known that in a dual-separating Fréchet space E any bounded sequence $(y_n) \subset E$ with $y_n \not\to 0$ such that $y_n \to 0$ weakly, contains a subsequence (y_{i_n}) which is a basic orthogonal sequence (see [1], Corollary 3.3). Unfortunately, in a Fréchet space E of countable type, if $(y_n) \subset E$ and $y_n \not\to 0$, then $y_n \not\to 0$ weakly ([4], Theorem 4.4 and Proposition 4.11). Using the idea of the proof of Corollary 3.3, [1], we show the following.

Proposition 4. *Let E be a Fréchet space and let (f_n) be a sequence of continuous functionals on E such that $\bigcap_{n=1}^{\infty} \ker f_n = \{0\}$. Then any bounded sequence $(y_n) \subset E$ with $y_n \not\to 0$ such that $\lim_n f_j(y_n) = 0$ for any $j \in \mathbb{N}$ contains a subsequence (y_{i_n}) which is a basic orthogonal sequence in E.*

Proof. By [1], Theorem 2.2 (see Introduction) it is enough to show that the bounded subset $Y = \{y_n : n \in \mathbb{N}\}$ of E is non-compactoid. Suppose, by contradiction, that Y is compactoid. Then the closed absolutely convex hull Z of Y is complete metrizable absolutely convex and compactoid. Let F be the linear span of (f_n). Since the Hausdorff locally convex topology $\sigma(E, F)$ on E is weaker than the original topology τ on E, then $\sigma(E, F)|Z = \tau|Z$ ([5], Theorem 3.2). Hence $y_n \to 0$, a contradiction. ∎

Corollary 5. *Let (x_n) be a sequence in a Banach space E. If there is a sequence (f_n) of continuous functionals on E with $\bigcap_{n=1}^{\infty} \ker f_n = \{0\}$ such that $f_n(x_m) = \delta_{n,m}$ for all $n, m \in \mathbb{N}$, then (x_n) contains a subsequence (x_{i_n}) which is a basic orthogonal sequence in E.*

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