Convexity and polynomial equations in Banach spaces with the Radon-Nikodym property

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Abstract

We define and characterize a new class of weakly compact subsets of spaces of integrable functions related to a new definition of polynomials in such spaces. We study the algebraic structure of ring of the set of all these polynomials and relate it to geometric and topological properties of subsets of $L_1(\mu)$. We use these results to study relative compactness and convexity of subsets of Banach spaces with the Radon-Nikodym property. An extension of the Uhl theorem about the range of a vector measure is obtained.

1 Introduction and notation.

In this paper we propose new tools for the study of the geometric and topological properties of the range of the integral operator defined by a (countably additive) vector measure of bounded variation. Let (Ω, Σ, μ) be a finite measure space and let $S(\mu)$ be the normed space of all the simple functions of $L_1(\Omega, \Sigma, \mu)$. Consider the (linear) space of polynomials $R[x]$. The procedure that we use consists on the definition of a ring structure with the elements of the tensor product $R[x] \otimes S(\mu)$. We call (Σ, μ)-polynomials the elements of this ring. If $\mathbb{P}$ is a (Σ, μ)-polynomial, we define a polynomial equation $\mathbb{P} = 0$ and find the set of its solutions in the space $L_1(\Omega, \Sigma, \mu)$, i.e., the set of the functions that satisfy $\mathbb{P}(f) = 0$ μ-almost everywhere when we substitute the variable x by the function f in $\mathbb{P}$. The main idea of this paper is to relate the properties of the ring $R[x] \otimes S(\mu)$ to the geometric and topological

Received by the editors January 1998- In revised form in July 2000.
Communicated by F. Bastin.
1991 *Mathematics Subject Classification :* Primary 46M05, 46A32.
Key words and phrases : Spaces of integrable functions, Banach spaces, Compactness, Convexity.

properties of the image of the integral operator defined by the vector measure F. In particular, we show that under certain conditions the image of the set of solutions of every polynomial equation is relatively compact and convex. This result may be considered as a generalization of the Uhl theorem about the compactness and convexity of the closure of the range of a vector measure (see section IX in [2]).

In section 2 we construct the algebraic structure of the ring $R[x] \otimes S(\mu)$ and we establish the basic properties of the sets of solutions of the polynomial equations $\mathbb{P} = 0$. The main result of Section 3 is the above mentioned generalization of the Uhl theorem (theorem 13). The rest of this section is devoted to the study of the relation between the ring product and the images of the sets of solutions of the equations as $\mathbb{P} = 0$. Finally, in section 4 we apply our technique to the study of the range of the operators from the space $L_1(\mu)$ into a Banach space with the Radon-Nikodym property E. To do so we first define two new classes of relatively weakly compact subsets of the spaces of integrable functions using sets of solutions of polynomial equations (see proposition 20). We obtain in this way several results about compact convex subsets of the range of an operator which are closely related to the polynomial structure of subsets of $L_1(\mu)$. The main conclusion of the last section is that compactness and convexity of subsets in the range of an operator can be characterized in terms of (Σ, μ)-polynomials (corollaries 21 and 22).

We use standard Banach space notation. We write "a.e" instead of "μ-almost everywhere" (or nothing if it is clear in the context). If B is a subset of a Banach space, we denote by $co(B)$ the convex hull of B. $R[x]$ is the linear space of all the polynomials as $P(x) = \sum_{i=0}^{n} \lambda_i x^i$, where $n \in \mathbf{N}$ and $\lambda_i \in \mathbf{K}$ ($\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$). We refer the reader to the chapter III of the book of W. Hungerford [4] for basic concepts and notation about rings and polynomials.

2 Definitions and algebraic results.

Definition 1. *We call to an element $\mathbb{P}(x) = \sum_{j=1}^{m} P_j(x) \otimes f_j \in R[x] \otimes S(\mu)$ -where $P_j \neq 0$ and $f_j \neq 0$- a (Σ, μ)-polynomial. Note that it is possible to find a broad class of representations of a (Σ, μ)-polynomial. In fact, each f_j is an equivalence class of functions that are equal a.e.. We define the maximum degree (the minimum degree) of a (Σ, μ)-polynomial $\mathbb{P}$ - max-deg$\mathbb{P}$ and min-deg$\mathbb{P}$ for short - as the minimum (the maximum) for all the representations of $\mathbb{P}$ of the maximum (the minimum) of the degrees of the polynomials $P_j(x)$ in the definition of $\mathbb{P}$.*

Definition 2. *Let $\mathbb{P} = \sum_{j=1}^{m} P_j(x) f_j \in R[x] \otimes S(\mu)$. We define the evaluation of $\mathbb{P}$ at the function $f \in S(\mu)$ as the function $\mathbb{P}(f) = \sum_{j=1}^{m} P_j(f) f_j$. We omit the straightforward proof of the fact that it does not depend on the particular representation of the tensor $\mathbb{P}$. We say that the expression $\mathbb{P} = 0$ is a (Σ, μ)-polynomial equation. A function $f \in L_1(\mu)$ is a solution of the equation $\mathbb{P} = 0$ if and only if $\mathbb{P}(f) = 0$.*

Let E be a Banach space, and let $T : S(\mu) \to E$ be a linear map. We also say that $z \in E$ is a T-solution of the (Σ, μ)-polynomial equation $\mathbb{P} = 0$ if and only if there exists a solution f of this equation such that $T(f) = z$.

Note that if $\mathbb{P}$ is a (Σ, μ)-polynomial, we can always find a representation of $\mathbb{P}$

as

$$\mathbb{P} = \sum_{j=1}^{m} P_j(x) \otimes \chi_{A_j},$$

where $A_j \in \Sigma$, $j = 1, ..., m$, χ_{A_j} is the characteristic function of A_j and $\mu(A_j \cap A_k) = 0$ if $j \neq k$. This fact simplifies the following definition.

Definition 3. *Let $\mathbb{P}, \mathbb{Q} \in R[x] \otimes S(\mu)$. If $\mathbb{P} = \sum_{j=1}^{m} P_j \otimes \chi_{A_j}$ and $\mathbb{Q} = \sum_{k=1}^{r} Q_k \otimes \chi_{B_k}$ are representations of $\mathbb{P}$ and $\mathbb{Q}$, we define the product $\mathbb{P}.\mathbb{Q}$ as*

$$\mathbb{P}.\mathbb{Q} = (\sum_{j=1}^{n} P_j \otimes \chi_{A_j}).(\sum_{k=1}^{r} Q_k \otimes \chi_{B_k}) = \sum_{j=1}^{n}\sum_{k=1}^{r} P_j Q_k \otimes \chi_{A_j \cap B_k}.$$

The proof of the fact that this product is well-defined is standard.

Proposition 4. *$(R[x] \otimes S(\mu), +, .)$ is a commutative ring with identity. However, it is not an integral domain.*

Proof. All the properties of the second operation can be easily checked. The identity is obviously the tensor $1 \otimes \chi_\Omega$. To see that there exist zero divisors in $R[x] \otimes S(\mu)$ it is enough to consider for instance the elements $1 \otimes \chi_A$ and $1 \otimes \chi_B$, where $A, B \in \Sigma$, $\mu(A \cap B) = 0$, $\mu(A) \neq 0$ and $\mu(B) \neq 0$. ■

In particular, this proposition means that $R[x] \otimes S(\mu)$ is not a unique factorization domain. However (non unique) factorizations of (Σ, μ)-polynomials will be relevant in section 4. It is easy to see that the definition of the maximum and minimum degrees of a (Σ, μ)-polynomial does not allow us to an Euclidean ring structure for $R[x] \otimes S(\mu)$. We use standard ring notation. Thus we say that $\mathbb{P}$ divides $\mathbb{Q}$ if there exists another (Σ, μ)-polynomial $\mathbb{H}$ such that $\mathbb{P}.\mathbb{H} = \mathbb{Q}$.

Proposition 5. *Let $f \in S(\mu)$. The map $T_f : R[x] \otimes S(\mu) \to S(\mu)$ given by*

$$T_f(\mathbb{P}) := \mathbb{P}(f), \qquad \mathbb{P} \in R[x] \otimes S(\mu)$$

is a homomorphism of rings.

The proof is standard.

Definition 6. *If S is a subset of $R[x] \otimes S(\mu)$, we define the annihilator of S as*

$$An(S) := \{f \in L_1(\mu) : \mathbb{P}(f) = 0 \ \text{a.e.}, \ \forall \mathbb{P} \in S\}.$$

In particular, if $\mathbb{P} \in R[x] \otimes S(\mu)$ and $S = \{\mathbb{P}\}$, we write $An(\mathbb{P})$ instead of $An(S)$. Thus $An(\mathbb{P})$ is the set of all the solutions of the (Σ, μ)-polynomial equation $\mathbb{P}(f) = 0$. If I is an ideal of $R[x] \otimes S(\mu)$ and $\mathbb{P} + I$ is an element of the quotient ring $\frac{R[x] \otimes S(\mu)}{I}$, we have

$$An(\mathbb{P} + I) := \{f \in L_1(\mu) : \mathbb{Q}(f) = 0 \ \text{a.e.} \ \forall \mathbb{Q} \in \mathbb{P} + I\}.$$

Remark 7. *A direct consequence of proposition 5 is that, if $\mathbb{P}$ divides $\mathbb{Q}$, then $An(\mathbb{P}) \subset An(\mathbb{Q})$. The converse is not in general true and we will come back to it later on. It is also easy to see that $An(\mathbb{P}) \cup An(\mathbb{Q}) \subset An(\mathbb{P}.\mathbb{Q})$.*

Proposition 8. *Let* $\mathbb{P} = \sum_{j=1}^{m} P_j \otimes S(\mu)$ *such that* $\{A_1, ..., A_m\}$ *are pairwise disjoint and* $\mu(A_j) \neq 0$ *for every* $j \in \{1, ..., m\}$. *Then*

i) $An(\mathbb{P}) \neq \varnothing$ *if and only if for each* $j \in \{1, ..., m\}$, P_j *can be written as* $P_j = (x - \lambda_j)Q_j$, *where* $Q_j \in R[x]$ *and* $\lambda_j \in \mathbf{K}$, *i.e. each* P_j *has at least a root in* $\mathbf{K}$.

ii) If $An(\mathbb{P}) \neq \varnothing$, *then* $\mathbb{P}$ *is not a zero divisor if and only if* $An(\mathbb{P})$ *is bounded (in* $L_1(\mu)$*).*

Proof. i) is obvious. For the proof of ii), let $A = \Cup_{j=1}^{n} A_j$. First, suppose that there is no function $f \in S(\mu)$ such that $f\chi_A$ satisfies $\mathbb{P}(f\chi_A) = 0$. Then $An(\mathbb{P}) = \varnothing$ and there is nothing to prove. However, if there exists such a function f it is obvious that for each $g \in S(\mu)$, $\mathbb{P}(f\chi_A + g\chi_B) = 0$, where $B = \Omega - A$. This means that $An(\mathbb{P})$ si not bounded if $\mu(B) \neq 0$. In this case 0 can be written as $\mathbb{P}.\mathbb{Q} = 0$, where $Q = 1 \otimes \chi_B \neq 0$, and then $\mathbb{P}$ is a zero divisor.

Now, if $\mu(B) = 0$ we can define the function $f_0 = \sum_{j=1}^{m} \nu_j \otimes \chi_{A_j}$, where the ν_j are defined as the roots of P_j which satisfy $|\nu_j| = \max\{|\lambda_j| : \lambda_j \textit{ is a root of } P_j\}$. On the one hand, it is easy to see that f_0 is a bound of the set $An(\mathbb{P})$ (with respect to the L_1 norm and the L_∞ norm). On the other hand, $\mathbb{P}.\mathbb{Q} = 0$ implies $\mathbb{Q} = 0$. Thus, if $\mu(B) = 0$, $\mathbb{P}$ is not a zero divisor and $An(\mathbb{P})$ is bounded. ■

Definition 9. *Let* $n \in \mathbf{N}$ *and* $f_1, ..., f_n \in S(\mu)$. *We say that* $f \in S(\mu)$ *is a* (Σ, μ)*-combination of* $f_1, ..., f_n$ *if there exist* $A_1, ..., A_n \in \Sigma$ *pairwise disjoint a.e. sets such that*

$$\Cup_{i=1}^{n} A_i = \Omega \ \textit{a.e.} \qquad \textit{and} \qquad f = \sum_{i=1}^{n} f_i \chi_{A_i} \ \textit{a.e.}.$$

We also define the set of all the (Σ, μ)*-combinations of these functions as*

$$C_{(\Sigma,\mu)}(f_1, ..., f_n) := \{f \in S(\mu) : f \textit{ is a } (\Sigma, \mu)\textit{-combination of } f_1, ..., f_n\}.$$

Proposition 10. *Let* $\mathbb{P} = \sum_{j=1}^{m} P_j \otimes S(\mu) \in R[x] \otimes S(\mu)$ *such that* $\{A_1, ..., A_,\}$ *are pairwise disjoint,* $\mu(A_j) \neq 0$ *and* $\Cup_{j=1}^{m} A_j = \Omega$ *a.e. . Suppose that for each* $j \in \{1, ..., m\}$ P_j *can be written as* $P_j = (x - \lambda_{1j})^{\alpha_{1j}}(x - \lambda_{2j})^{\alpha_{2j}} ...(x - \lambda_{n_j j})^{\alpha_{n_j j}} Q_j(x)$, *where* Q_j *can not be factored in* $\mathbf{K}$, $\lambda_{kj} \neq \lambda_{rj}$ *iff* $k \neq r$, $\alpha_{kj} \geq 1$ *and* $n_j \geq 1$. *Then there exist* $f_1, ..., f_n \in An(\mathbb{P})$ *such that* $C_{(\Sigma,\mu)}(f_1, ..., f_n) = An(\mathbf{P})$, *where* $n = max\{n_j : j = 1, ..., m\}$.

Proof. First, it is obvious that the annihilator of $\mathbb{P}$ coincides with the annihilator of the (Σ, μ)-polynomial $\mathbb{Q}$ that satisfies the same factorization with $\alpha_{kj} = 1$ for every k, j. Then we suppose $\alpha_{kj} = 1$ $\forall k, j$. Let us define the functions

$$f_i = \sum_{j=1}^{m} \lambda_{ij} \chi_{A_j}, \quad i = 1, ..., n,$$

where $\lambda_{ij} = \lambda_{n_j j}$ for each j if $i \geq n_j$. Then

$$\mathbb{P}(f_i) = \sum_{i=1}^{m} P_j(f_i)\chi_{A_j} = \sum_{j=1}^{m} (f_i - \lambda_{1j})(f_i - \lambda_{2j})...(f_i - \lambda_{n_j j}) Q_j(f_i) \chi_{A_j} = 0,$$

and $f_i \in An(\mathbb{P})$ for every i.

Now, suppose that $f \in C_{(\Sigma,\mu)}$. There exist pairwise disjoint subsets $B_i \in \Sigma$ that satisfy all the conditions of the definition 9, and $f = \sum_{i=1}^{n} f_i \chi_{B_i}$. Then

$$\mathbb{P}(f) = \sum_{j=1}^{m} P_j(f)\chi_{A_j} = \sum_{j=1}^{m} P_j(\sum_{i=1}^{n} f_i \chi_{B_i})\chi_{A_j} = \sum_{j=1}^{m}\sum_{i=1}^{n} P_j(f_i)\chi_{B_i \cap A_j} = 0.$$

Conversely, suppose that $f \in An(\mathbb{P})$. Then $P_j(f\chi_{A_j}) = 0$ for each j. Thus,

$$f\chi_{A_j} = \sum_{i=1}^{n_j} \lambda_{ij}\chi_{B_{ij}},$$

where B_{ij} are a.e. pairwise disjoint subsets for A_j, and $\Cup_{i=1}^{n_j} B_{ij} = A_j$ a.e.. Let us define $B_i = \Cup_{j=1}^{m} B_{ij}$ for $i = 1, ..., n$, where $B_{ij} = \varnothing$ if $i > n_j$. Therefore,

$$f = \sum_{j=1}^{m}\sum_{i=1}^{n_j} \lambda_{ij}\chi_{B_{ij}} = \sum_{j=1}^{m}\sum_{i=1}^{n_j} f_i\chi_{B_{ij}} = \sum_{i=1}^{n}\sum_{j=1}^{m} f_i\chi_{B_{ij}} = \sum_{i=1}^{n} f_i\chi_{B_i},$$

and we have also the inclusion $An(\mathbb{P}) \subset C_{(\Sigma,\mu)}(f_1, ..., f_n)$. ∎

In section 3 we will use the factorable reduced (Σ, μ)-polynomial $\mathbb{Q}$ associated to $\mathbb{P}$ defined as in the begining of the former proof, but also satisfying that each Q_j has degree 0. It is obvious that even in this case $An(\mathbb{P}) = An(\mathbb{Q})$.

Proposition 11. *Let $\mathbb{P}, \mathbb{Q} \in R[x] \otimes S(\mu)$. Consider the principal ideal $I_{\mathbb{P}}$ generated by $\mathbb{P}$ and the quotient ideal $\frac{R[x]\otimes S(\mu)}{I_{\mathbb{P}}}$. Then*

$$An(\mathbb{Q} + I_{\mathbb{P}}) = An(\mathbb{Q}) \cap An(\mathbb{P}).$$

Proof. Let $f \in An(\mathbb{Q}+I_{\mathbb{P}})$. Since $0 \in I_{\mathbb{P}}$, we get $\mathbb{Q}(f) = (\mathbb{Q}+0)(f) = 0$. Then if we take $1 \otimes \chi_\Omega \in R[x] \otimes S(\mu)$ we have $(\mathbb{Q} + 1 \otimes \chi_\Omega.\mathbb{P})(f) = \mathbb{Q}(f) + \mathbb{P}(f) = 0$. Thus $\mathbb{P}(f) = 0$. Now suppose that $f \in An(\mathbb{Q}) \cap An(\mathbb{P})$ and consider $\mathbb{Q} + \mathbb{H}\mathbb{P} \in \frac{R[x]\otimes S(\mu)}{I_{\mathbb{P}}}$. Then

$$(\mathbb{Q} + \mathbb{H}\mathbb{P})(f) = \mathbb{Q}(f) + \mathbb{H}(f)\mathbb{P}(f) = 0.$$

∎

Proposition 12. *Let $\mathbb{P} \in R[x] \otimes S(\mu)$ and let $f \in An(\mathbb{P})$. If $\mathbb{P}$ is not a zero divisor, then there exists a (Σ, μ)-polynomial $\mathbb{P}_f$ with max-deg=1 such that $\mathbb{P}_f$ divides $\mathbb{P}$ and $An(\mathbb{P}_f) = \{f\}$.*

Proof. Let $\mathbb{P} \in R[x] \otimes S(\mu)$ and let $\sum_{j=1}^{m} P_j \otimes \chi_{A_j}$ be a representation of $\mathbb{P}$ such that $\{A_1, ..., A_m\}$ are pairwise disjoint, $\mu(A_j) \neq 0$ for each j and $\Cup_{j=1}^{m} A_j = \Omega$ a.e.. Consider the factorization of $\mathbb{P}$ given in proposition 10. Since $f \in An(\mathbb{P})$, an application of proposition 10 allows us to write f as $f = \sum_{i=1}^{n} f_i\chi_{B_i}$, where $\Cup_{i=1}^{n} B_i = \Omega$. The f_i $(i = 1, ..., n)$ can be written as $f_i = \sum_{j=1}^{m} \lambda_{ij}\chi_{A_j}$, following the notation of proposition 10. Then we get $f = \sum_{i=1}^{n}(\sum_{j=1}^{m} \lambda_{ij}\chi_{A_j})\chi_{B_i}$. Now let us define the (Σ, μ)-polynomial $\mathbb{P}_f$ as

$$\mathbb{P}_f = \sum_{i=1}^{n}\sum_{j=1}^{m}(x - \lambda_{ij})\chi_{A_j \cap B_i}.$$

It is easy to see that $An(\mathbb{P}_f) = \{f\}$. Moreover, if we define a (Σ, μ)-polynomial $\mathbb{H}$ as $\mathbb{H} = \sum_{i=1}^{n}\sum_{j=1}^{m} H_{ij}\chi_{A_j \cap B_i}$ - where $H_{ij}(x - \lambda_{ij}) = P_j$ for each $j = 1, ..., m$ in the decomposition given in 10 - a straightforward calculation shows that $\mathbb{P}_f\mathbb{H} = \mathbb{P}$. ∎

3 Convexity and (Σ, μ)-polynomials.

If E is a Banach space, let $F : \Sigma \to E$ be a countably additive vector measure defined on a σ field Σ, and let μ a (countably additive) measure such that $F << \mu$ (μ is a control measure for F). In this case the integration map $T_F : L_\infty(\mu) \to E$ given by $T_F(f) := \int f dF$ is a well-defined weak* to weak continuous linear operator (see Corollary I.2.7 in [2]). Let us consider the restriction of this map to $S(\mu)$. The aim of this section is to study how the structure of the annihilators of (Σ, μ)-polynomials concerns to the range of this map. The main result of this section is the following theorem, which can be considered as a generalization of the Uhl theorem. In the rest of this section we will assume that μ is a (nonatomic) finite measure.

Theorem 13. *Let E be a Banach space with the Radon-Nikodym property. Let $\mathbb{P}$ be a (Σ, μ)-polynomial such that it is not a zero divisor. Let $F : \Sigma \to E$ be a countably additive vector measure which is of bounded variation and nonatomic and $F << \mu$. Let $T_F : S(\mu) \to E$ be the integration map $T_F(f) := \int f dF$. Then the norm closure of $T_F(An(\mathbb{P}))$ is convex and norm compact.*

Proof. First, suppose that max-deg$\mathbb{P} \leq 2$. If either $An(\mathbb{P}) = \varnothing$ or $An(\mathbb{P}) = \{f_0\}$ there is nothing to prove (these are the cases if max-deg$\mathbb{P} \leq 1$). Suppose that max-deg$\mathbb{P} = 2$. Then it is easy to see that an application of proposition 10 allows us to write $An(\mathbb{P})$ as

$$An(\mathbb{P}) = \{f_1\chi_A + f_2\chi_B : A, B \in \Sigma, A \cup B = \Omega \; a.e., \, and \; A \cap B = \varnothing \; a.e.\},$$

where f_1 and f_2 are solutions of $\mathbb{P}$. Put $f_3 = f_2 - f_1 \in S(\mu)$. We can also write each $f \in An(\mathbb{P})$ as

$$f = f_1\chi_{(\Omega - B)} + f_2\chi_B = f_1 + (f_2 - f_1)\chi_B = f_1 + f_3\chi_B,$$

and then

$$An(\mathbb{P}) = \{f_1 + f_3\chi_B : B \in \Sigma\}.$$

Now, if we denote x_1 to the element $\int f_1 dF$ we can write each $x \in T_F(An(\mathbb{P}))$ as

$$x = T_F(f) = \int f_1 dF + \int f_3\chi_B dF = X_1 + \int f_3\chi_B dF$$

where $B \in \Sigma$. An appeal to the argument given in the proof of theorem 10 in the section IX.1 of [2] shows that the norm closure of the set

$$\{\int f_3\chi_B dF : B \in \Sigma\}$$

is compact and convex, and then the same is true for the set $\overline{T_F(An(\mathbb{P}))}$.

Now suppose that max-deg$\mathbb{P} > 2$. If $An(\mathbb{P}) = \varnothing$ there is nothing to prove. Suppose that $An(\mathbb{P}) \neq \varnothing$. Since $\mathbb{P}$ is not a zero divisor an application of proposition I.8 gives that $An(\mathbb{P})$ is bounded. Is is easy to see that the map $T_F : L_\infty(\mu) \to E$ is a compact operator (see for example the proof of theorem 10 in section IX.1 in [2]) and then $T_F(An(\mathbb{P}))$ is relatively compact in E. To prove that the norm closure of this set is convex, let x_1 and x_2 be in the closure of $T_F(An(\mathbb{P}))$. Let $0 \leq \alpha \leq 1$ and $\epsilon > 0$, and choose $f_1, f_2 \in An(\mathbb{P})$ such that

$$\|x_i - T_F(f_i)\| < \frac{\epsilon}{2} \quad for \quad i = 1, 2.$$

Let us define $\mathbb{P}_{f_1}$ and $\mathbb{P}_{f_2}$ as in proposition 12. Consider the reduced (Σ, μ)-polynomial $\mathbb{Q}$ associated to $\mathbb{P}_{f_1}.\mathbb{P}_{f_2}$ obtained as in the begining of the proof of the proposition 10. The annihilator of $\mathbb{Q}$ satisfies $An(\mathbb{P}_{f_1}.\mathbb{P}_{f_2}) = An(\mathbb{Q})$. It is easy to see that $\mathbb{Q}$ divides $\mathbb{P}$. Moreover, its degree is ≤ 2, and then we know that $\overline{T_F(An(\mathbb{P}_{f_1}.\mathbb{P}_{f_2}))}$ is compact and convex. On one hand,

$$\|\alpha x_1 + (1-\alpha)x_2 - (\alpha T_F(f_1) + (1-\alpha)T_F(f_2))\| \leq$$

$$\leq \alpha\|x_1 - T_F(f_1)\| + (1-\alpha)\|x_2 - T_F(f_2)\| < \epsilon.$$

On the other hand, an appeal to remark 7 shows that

$$\overline{T_F(An(\mathbb{P}_{f_1}.\mathbb{P}_{f_2}))} \subset \overline{T_F(An(\mathbb{P}))}.$$

Thus for each $\epsilon > 0$ there is an element $x_\epsilon := \alpha T_F(f_1)(1-\alpha)T_F(f_2) \in \overline{T_F(An(\mathbb{P}))}$ such that $\|(\alpha x_1 + (1-\alpha)x_2) - x_\epsilon\| < \epsilon$. This completes the proof. ■

Theorem 13 and remark 7 give us information about the structure of the set of T_F-solutions of the (Σ, μ)-polynomial equation $\mathbb{P} = 0$ related to the set of divisors of $\mathbb{P}$. If $\mathbb{Q}$ divides $\mathbb{P}$ then the closure of the set of T_F-solutions of $\mathbb{Q} = 0$ is a convex and compact subset of the closure of the set of T_F-solutions of $\mathbb{P} = 0$. This fact is obvious for the set of T_f-solutions of the (Σ, μ)-polynomial equation $(x-1)x \otimes \chi_\Omega = 0$, which is exactly the range of the vector measure F. In the following we obtain several results about the structure of the sets of T_F-solutions using the quotient ring $\frac{R[x]\otimes S(\mu)}{I_{\mathbb{P}}}$ and proposition 12. Throughout this section let $\mathbb{P}$ be a (non zero divisor) (Σ, μ)-polynomial, let E be a Banach space with the Radon-Nikodym property and let $F : \Sigma \to E$ be a countably additive vector measure satisfying $F << \mu$ which is of bounded variation and nonatomic.

Definition 14. *We define the class $G_{\mathbb{P},F}$ of compact subsets of $G := \overline{T_F(An(\mathbb{P}))} \subset E$ as*

$$G_{\mathbb{P},F} = \{\overline{T_F(An(Q + I_{\mathbb{P}}))} : \mathbb{Q} \in R[x] \otimes S(\mu)\}.$$

We denote these sets by $G_{\mathbb{P},F}(\mathbb{Q}) := \overline{T_F(An(\mathbb{Q} + I_{\mathbb{P}}))}$. We also define the binary operation $* : G_{\mathbb{P},F} \times G_{\mathbb{P},F} \to G_{\mathbb{P},F}$ as

$$G_{\mathbb{P},F}(\mathbb{Q}) * G_{\mathbb{P},F}(\mathbb{H}) = G_{\mathbb{P},F}(\mathbb{Q}.\mathbb{H}).$$

Proposition 11 and the definition of the product in the quotient give that $*$ is well-defined. Note that all the elements of $G_{\mathbb{P},F}$ are convex sets too, since we can always find a (Σ, μ)-polynomial that is not a zero divisor belonging to each class $\mathbb{Q} + I_{\mathbb{P}}$.

Proposition 15. *The map $J : \frac{R[x]\otimes S(\mu)}{I_{\mathbb{P}}} \to G_{\mathbb{P},F}$ biven by*

$$J(\mathbb{Q} + I_{\mathbb{P}}) := G_{\mathbb{P},F}(\mathbb{Q})$$

is an epimorphism between the (multiplicative) structure $(\frac{R[x]\otimes S(\mu)}{I_{\mathbb{P}}},.)$ *and the abelian monoid* $(G_{\mathbb{P},F},*)$. *Moreover,* $(G_{\mathbb{P},F},*)$ *satisfies:*

a) $\forall A\in G_{\mathbb{P},F}$, $A*\varnothing=A$, $A*A=A$ *and* $A*G=G$.

b) $\forall A,B\in G_{\mathbb{P},F}$, $co(A\cup B)\subset A*B$.

Proof. It is easy to see that the elements $G_{\mathbb{P},F}(\mathbb{Q})$ and the operation $*$ are actually defined using the fact that the map J is an epimorphism. Moreover, the image of the identity $1\otimes\chi_\Omega+I_{\mathbb{P}}$ is $\varnothing$, since an application of proposition 11 gives

$$An(1\otimes\chi_\Omega+I_{\mathbb{P}})=\varnothing\cap An(\mathbb{P})=\varnothing$$

and then $A*\varnothing=A$ for every $A\in G_{\mathbb{P},F}$. Since for each $\mathbb{Q}$, $\mathbb{Q}.\mathbb{P}\in I_{\mathbb{P}}$, we also have $A*G=G$ for every $A\in G_{\mathbb{P},F}$. Since $An(\mathbb{P})=An(\mathbb{P}^2)$ the equality $A*A=A$ is obvious. b) is just a consequence of the theorem 13 and the remark 7. ■

The following results are -from the algebraic point of view- just straightforward consequences of the structure of the zero divisors of the quotient ring $\frac{R[x]\otimes S(\mu)}{I_{\mathbb{P}}}$.

Corollary 16. *If* $\mathbb{P}$ *and* $\mathbb{Q}$ *are factorable reduced* (Σ,μ)*-polynomials such that* $\mathbb{Q}$ *divides* $\mathbb{P}$, *min-deg*$\mathbb{Q}$*=n and max-deg*$\mathbb{P}$*=m, then there exists a* (Σ,μ)*-polynomial* $\mathbb{H}$ *such that max-deg*$\mathbb{H}\le(m-n)$ *and* $\mathbb{Q}.\mathbb{H}=\mathbb{P}$, *and then* $G_{\mathbb{P},F}(\mathbb{Q})*G_{\mathbb{P},F}(\mathbb{H})=G$.

Corollary 17. *Let* $\mathbb{P}$ *be a factorable reduced* (Σ,μ)*-polynomial such that max-deg*$\mathbb{P}$*=n. If* $f\in An(\mathbb{P})$ *then there exist* (Σ,μ)*-polynomials* $\mathbb{P}_i$, $i=1,...,n$, *such that*

1) $An(\mathbb{P}_1)=\{f\}$.

2) max-deg$\mathbb{P}_i$*=1,* $i=1,...,n$.

3) $G=G_{\mathbb{P},F}(\mathbb{P}_1)*...*G_{\mathbb{P},F}(\mathbb{P}_n)$.

Proof. It is enough to consider the polynomial $\mathbb{P}_f$ given in proposition 12 and to apply n times the former corollary. ■

4 Applications. The polynomial structure of the range of an operator on L_1.

In this section we get some topological properties of the subsets of $L_1(\mu)$ of the type $An(\mathbb{P})$. It is easy to see -and it has been said in section 2 - that for each $\mathbb{P}$, $An(\mathbb{P})$ is a subset of $S(\mu)$, although it has been defined as a subset of $L_1(\mu)$. Our aim is to apply the results of sections 2 and 3 in order to study how the "polynomial structure" of $L_1(\mu)$ concerns to the range of an operator $T:L_1(\mu)\to E$, where E is a Banach space with the Radon-Nikodym property. Since the function $F:\Sigma\to E$ given by $F(A):=T(\chi_A)$ is a countably additive μ-continuous vector measure of bounded variation, (see theorem 5 in the chapter III.1 of [2]) all the results of section 3 about vector measures can be applied to $T(An(\mathbb{P}))$.

Definition 18. *Let* $K\subset L_1(\mu)$. *We say that* K *is approximable by polynomials if for each* $\epsilon>0$ *there exists* $\mathbb{P}\in R[x]\otimes S(\mu)$ *that is not a zero divisor such that*

$$\forall f\in K \quad \text{there exists} \quad f_0\in An(\mathbb{P}):\ \|f-f_0\|<\epsilon. \qquad (1)$$

We also say that K is strictly approximable by polynomials if $\forall\epsilon > 0$ there is a (Σ,μ)-polynomial $\mathbb{P}$ that is not a zero divisor satisfying (1) and

$$\forall f_0 \in An(\mathbb{P}) \quad \text{there exists} \quad f \in K: \ \|f - f_0\| < \epsilon. \tag{2}$$

Lemma 19. *Let $f \in S(\mu)$. Then there exists a (Σ,μ)-polynomial $\mathbb{P}_f$ such that $An(\mathbb{P}_f) = \{f\}$.*

Proof. If $f = \sum_{i=1}^m \lambda_i \chi_{A_i}$, (where $\{A_i\}_{i=1}^n$ defines a partition of Ω) it is enough to define $\mathbb{P}_f$ as $\mathbb{P}_f := \sum_{i=1}^m (x - \lambda_i) \otimes \chi_{A_i}$. ■

Proposition 20. *a) If $K \subset L_1(\mu)$ is compact, then it is approximable by polynomials.*

b) If $K \subset L_1(\mu)$ is approximable by polynomials, then it is relatively weakly compact.

Proof. a) Let $\epsilon > 0$. Take the open covering of K given by the sets of open balls $\{B_{\frac{\epsilon}{2}}(f) : f \in K\}$. Then there is a finite set $f_1, ..., f_n \in K$ such that $K \subset \Cup_{i=1}^n B_{\frac{\epsilon}{2}}(f_i)$. Since the set $S(\mu)$ is dense in $L_1(\mu)$, there are n step functions $f_{1,0}, ..., f_{n,0}$ such that $\|f_i - f_{i,0}\| < \frac{\epsilon}{2}$ for every i. We known that for each $i = 1, ..., n$ there exists a polynomial $\mathbb{P}_i$ such that $f_{i,0} \in An(\mathbb{P}_i)$ (lemma 19). Moreover, if $\mathbb{P} = \Pi_{i=1}^n \mathbb{P}_i$ then $f_{i,0} \in An(\mathbb{P})$ for each index i. Now, for every $f \in K$ there is an index $i \in \{1, ..., n\}$ such that $f \in B_{\frac{\epsilon}{2}}(f_i)$, and then the inequalities

$$\|f - f_{i,0}\| \le \|f - f_i\| + \|f_i - f_{i,0}\| < \epsilon$$

give the result.

b) We know by Dunford Theorem (see theorem III2.15 of [2] or theorem 8.9 of [3]) that it is enough to prove that K is bounded and uniformly integrable. Let $\epsilon > 0$ and let $\mathbb{P}$ satisfying the conditions of the definition 18. As a consequence of proposition 8.ii), we get that there is a constant $c > 0$ such that for every $f_0 \in An(\mathbb{P})$, $\|f_0\| \le c$, and then $\|f\| < c + \epsilon$ for every $f \in K$. Thus K is bounded.

Now we prove that K is uniformly integrable. Let $\epsilon > 0$. Consider a (non zero divisor) (Σ,μ)-polynomial $\mathbb{P}$ that satisfies the conditions of the definition 18 for $\frac{\epsilon}{2}$. It is easy to see that $An(\mathbb{P})$ is also bounded in $L_\infty(\mu)$. Then there exists a constant $c > 0$ such that

$$\forall f_0 \in An(\mathbb{P}), \qquad |f_0| \le c\chi_\Omega \ \ a.e..$$

Consider a sequence of measurable subsets $\{E_n\}$ such that $\mu(E_n) \to 0$. Then, if $f \in K$ and $\mu(E_n) < \frac{\epsilon}{2c}$, the following inequalities give the result.

$$\int_{E_n} |f| d\mu \le \int_{E_n} |f_0| d\mu + \int_{E_n} |f - f_0| d\mu \le c\mu(E_n) + \int_{E_n} |f - f_0| d\mu <$$

$$< \frac{\epsilon}{2} + \int_\Omega |f - f_0| d\mu < \epsilon.$$

■

Corollary 21. *Let $K \subset L_1(\mu)$ be approximable by polynomials and $T : L_1(\mu) \to E$ be an operator. Then $T(K)$ is relatively compact. Moreover, if K is strictly approximable by polynomials, $\overline{T(K)}$ is convex too.*

Proof. The first statement is a consequence of the properties of representable operators from $L_1(\mu)$ (see [2]) and proposition 20. For the second one, take $\epsilon > 0$ and a (Σ, μ)-polynomial $\mathbb{P}$ satisfying the conditions of the definition for $\frac{\epsilon}{3\|T\|}$. If $0 \leq \alpha \leq 1$ and $f_1, f_2 \in K$, then there are functions $f_{10}, f_{20} \in An(\mathbb{P})$ such that $\|f_i - f_{i0}\| < \frac{\epsilon}{3\|T\|}$, $i = 1, 2$. Since $\overline{T(An(\mathbb{P}))}$ is convex (see theorem 13) there is a function $f_0 \in An(\mathbb{P})$ such that

$$\|\alpha T(f_{10}) + (1-\alpha)T(f_{20}) - T(f_0)\| < \frac{\epsilon}{3},$$

and a function $f_\alpha \in K$ such that $\|f_\alpha - f_0\| < \frac{\epsilon}{3\|T\|}$. Thus

$$\|\alpha T(f_1) + (1-\alpha)T(f_2) - T(f_\alpha)\| \leq$$

$$\leq \|\alpha T(f_1) - \alpha T(f_{10})\| + \|(1-\alpha)T(f_2) - (1-\alpha)T(f_{20}) - T(f_0)\| + \|T(f_0) - T(f_\alpha)\| +$$

$$+ \|\alpha T(f_{10}) + (1-\alpha)T(f_{20}) - T(f_0)\| \leq$$

$$\leq \alpha\|T\|\|f_1 - f_{10}\| + (1-\alpha)\|T\|\|f_2 - f_{20}\| + \|T\|\|f_0 - f_\alpha\| + \frac{\epsilon}{3} < \epsilon.$$

■

It is well-known that if A is a relatively weakly compact set, then $\overline{co(A)}$ is weakly compact (see e.g. corollary 8 in section II.C of [7]). In particular, if A is a relatively weakly compact subset of $L_1(\mu)$, $T(\overline{co(A)})$ is a relatively compact convex set (see e.g. the (Dunford-Pettis) lemma 11 in chapter 19 of [2]). Our results give more information about these sets. Corollary 21 means that the norm closure of $T(\overline{co(A)})$ -where K is a strictly approximable by polynomials set- is the compact convex set $\overline{T(K)}$.

Corollary 22. *Let $B \subset T(L_1(\mu))$ be a relatively compact set and let $\epsilon > 0$. Then there is a (Σ, μ)-polynomial $\mathbb{P}_\epsilon$ such that $\forall x \in \overline{co(B)}$ there exists a function $f_0 \in An(\mathbb{P}_\epsilon)$ satisfying $\|T(f_0) - x\| < \epsilon$.*

Proof. Since $\overline{co(B)}$ is compact we can find a finite subcover $B_{\frac{\epsilon}{2}}(x_1), ..., B_{\frac{\epsilon}{2}}(x_n)$, $x_1, ..., x_n \in \overline{co(B)}$. Let $f_i \in L_1(\mu)$ such that for each $i = 1, ..., n$, $T(f_i) = x_i$. We can take n functions $f_{10}, ..., f_{n0} \in S(\mu)$ such that $\|f_i - f_{i0}\| < \frac{\epsilon}{2}$. Then it is enough to define $\mathbb{P}$ as the product $\mathbb{P} = \mathbb{P}_{f_{10}} ... \mathbb{P}_{f_{n0}}$, where $\mathbb{P}_{f_{i0}}$ are the (Σ, μ)-polynomials associated to f_{i0} given in lemma 19. ■

All the structure results given in section 2 and 3 can be applied to $T(An(\mathbb{P}))$ and directly extended to the class of the approximable by polynomials sets. After corollary 22 we know that we have a polynomial structure "close to" each convex compact subset of a Banach space with the Radon-Nikodym property. Next corollary relates this structure to the set of extreme points of $\overline{T(K)}$, as an application of the Krein-Milman theorem (see e.g. theorem 3.21 of [6] or theorem 3.3 in chapter VII of [2]).

Corollary 23. *Let $\epsilon > 0$. Then the (Σ, μ)-polynomial $\mathbb{P}_\epsilon$ obtained in corollary 22 can be written as a product $\mathbb{P}_\epsilon = \mathbb{P}_1.\mathbb{P}_2...\mathbb{P}_{n_\epsilon}$, $n_\epsilon \in \mathbf{N}$, where each $\mathbb{P}_i$, $i = 1, ..., n_\epsilon$ satisfies*

1) $An(\mathbb{P}_i) = \{f_i\}$.

2) $T(f_i)$ is an extreme point of $\overline{co(B)}$.

The technique and the results of this paper would be extended in several ways. In particular, we have the following open problems. On the one hand, the -rather strong- restrictions that we have imposed to the vector measures may be weakened using for instance the theory obtained by C. Muscalu in [5]. In this paper he generalizes several classical results about compactness and convexity of the range of a vector measure (Liapounoff theorem, Uhl theorem) in the case when the control measure μ is σ-finite. On the other hand, we may consider Radon-Nikodym sets in general Banach spaces instead of spaces with the Radon-Nikodym property (see [1]).

Another open problem which is closely related to the technique that we have shown is the following. Is it possible to find a (locally convex) topology for the tensor product $R[x] \otimes S(\mu)$ such that the class of the annihilators of its elements coincides with the class of the relatively weakly compact sets?.

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