# On the interchange of series and some applications

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**Abstract**

In this paper we prove the basic matrix theorem of Antosik-Swartz under weaker hypotheses than the ones they used. We obtain the converse result for complete normed spaces and generalize Antosik's interchange theorem for double series in a normed space. As a consequence, a number of characterizations on convergence in several spaces of vector sequences are derived. Finally, we obtain a version of the Orlicz-Pettis theorem for Banach spaces with a Schauder basis.

## 1 Introduction

Let $(x_{ij})_{i,j}$ be a matrix in a normed space $X$ such that: (1) For each $j \in \mathbb{N}$, the sequence $(x_{ij})_i$ is convergent to some $x_j \in X$ and (2) For each infinite set $M \subset \mathbb{N}$ there exists an infinite set $P \subseteq M$ such that $\left(\sum_{j\in P} x_{ij}\right)_i$ is a Cauchy sequence. In this setting, the Basic Matrix Theorem ([3]) asserts that the sequences $(x_{ij})_i$ are uniformly convergent on $j \in \mathbb{N}$ and $(x_j)_j$ converges to zero.

Many applications of the Basic Matrix Theorem in measure theory and Banach spaces have been found since its appearance ([3], [14]), such as generalizations of the uniform boundedness principle, the Banach-Steinhaus theorem and the classical Schur and Phillips lemmas. It should also be noticed that the usual conditions of the Basic Matrix theorem imply that every row $\overline{x}^i = (x_{ij})_j$ is an element of $c_0(X)$

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(the space of null sequences in $X$ endowed with the supremum norm) and that this theorem gives a sufficient condition for the convergence of a sequence in $c_0(X)$.

Florencio, Paúl and Virués [7] established an improvement of the Mikusiński-Antosik diagonal theorem for topological groups, dealing with the behavior of the diagonal of certain infinite matrices whose elements belong to a topological group. This improvement yields both the Basic Matrix Theorem and the Antosik diagonal lemma ([3]). Fleischer [6] generalized these two results for $K$-convergence in topological groups by using "hump" techniques. One of the basic ideas in that generalization is the following: if $A$ is an infinite matrix in a $K$-space such that for some infinite index set the rows converge to zero, then it is possible to obtain an infinite subset for which these rows are absolutely summable. Traynor [13] generalized the Mikusiński-Antosik theorem to noncommutative topological groups, and also gave some applications to the Schur lemma, the Nikodym convergence theorem, the Phillips lemma and the Banach-Steinhaus theorem.

In this paper we improve the Basic Matrix Theorem by using separation properties of natural families, those subfamilies of $P(\mathbb{N})$ which contain the finite subsets, with weaker hypotheses than those mentioned at the beginning of this introduction. For complete spaces, the converse result will also be proved and, as a corollary, we will obtain a characterization of convergence in $c_0(X)$.

A result of Swartz's [12] that generalizes the Antosik interchange theorem ([2]) will be improved in Theorem 3.6. The technique we will use in the characterization of convergence in $c_0(X)$ can be partially translated to $cs(X)$, the space of convergent series in $X$ endowed with the norm $\|(x_i)_i\| = \sup_n \|\sum_{i=1}^n x_i\|$. This technique, together with the improvement of the Swartz result, will allow us to obtain a characterization of the convergence of sequences in $cs(X)$. We can also consider other characterization from the isomorphism between $c_0(X)$ and $cs(X)$ (see Remark 3.9). Analysis similar to that allows us to study two isomorphisms, the one from $c(X)$, the space of convergent sequences in $X$ endowed with the supremum norm, to $cs(X)$ and the other one from $c(X)$ to $c_0(X)$, in order to extend the previous characterizations to the space $c(X)$ (see Section 4).

Our results will be applied to Banach spaces with a Schauder basis, obtaining a characterization of convergence and unconditional convergence of series by means of the weak topology $\sigma(X, M)$, where $M$ is the basic sequence in $X^*$ associated with the given Schauder basis. Therefore, we will also obtain a new version of the Orlicz-Pettis theorem and observe that this result generalizes another result of Swartz's [11].

Although this paper has been developed within the framework of normed space theory, most of the results could be extended, with some precautions, to normed groups by using the techniques followed in [10] and [14]. As the topology of any topological group is always generated by a family of quasi-norms ([4]), our results could also be extended to topological groups.

## 2 The Basic Matrix Theorem: convergence in $c_0(X)$

DEFINITION 2.1. *We say that $\mathcal{F}$ is a natural family if $\phi_0(\mathbb{N}) \subseteq \mathcal{F} \subseteq P(\mathbb{N})$, where $\phi_0(\mathbb{N})$ denotes the family of finite subsets of $\mathbb{N}$.*

*Let $\mathcal{F}$ be a natural family and let $\sum_{i\geq 1} x_i$ be a series in the normed space $X$. We say that the series $\sum_i x_i$ is $\mathcal{F}$-convergent (resp. $\mathcal{F}$-Cauchy, $\mathcal{F}$-weakly convergent, $\mathcal{F}$-weakly Cauchy) if $\sum_{i\in A} x_i$ is convergent (resp. Cauchy, weakly convergent, weakly Cauchy), for each $A \in \mathcal{F}$.*

*It is said that a natural family $\mathcal{F}$ has property SC if for every infinite set $M \subseteq \mathbb{N}$ there exists an infinite set $P \subseteq M$ such that $P \in \mathcal{F}$.*

In the literature, a family with property SC is also called a permeating family ([9]).

The basic matrix theorem implies that, if $\mathcal{F}$ is a natural family with property $SC$ and $(x_{ij})_{i,j}$ is a matrix in a Banach space $X$ such that $\left(\sum_{j\in B} x_{ij}\right)_i$ is a convergent sequence for each $B \in \mathcal{F}$, then the sequences $(x_{ij})_i$ are uniformly convergent on $j \in \mathbb{N}$.

DEFINITION 2.2. *We say that a natural family $\mathcal{F}$ has property $P_{c_0}$ if there exists a map $f : \mathbb{N} \to \mathbb{N}$ such that for every pair of sequences $(j_r)_r$ and $(m_r)_r$ in $\mathbb{N}$ with $j_1 < m_1 < j_2 < m_2 < \cdots$ there exists an infinite set $M \subseteq \mathbb{N}$ and $B \in \mathcal{F}$ that verify:*

*(a) $(m_{r-1}, m_r) \cap B = \{j_r\}$, for each $r \in M$.*

*(b) $card([m_{r-1}, m_r] \cap B) \leq f(r)$, for each $r \in \mathbb{N} \setminus M$.*

It is easily seen that each natural family with property SC also has property $P_{c_0}$; however, it will be shown that there exist natural families which have property $P_{c_0}$ and lack property SC (see remark at the end of this section).
We now prove that the previous result remains valid for natural families with property $P_{c_0}$.

THEOREM 2.3. *Let $(x_{ij})_{i,j}$ be a matrix in the normed space $X$ with the following properties:*

1. *For each $j \in \mathbb{N}$, $(x_{ij})_i$ is a Cauchy sequence.*
2. *There exists a map $f : \mathbb{N} \to \mathbb{N}$ such that if $(j_r)_r$ and $(m_r)_r$ are sequences of natural numbers with $j_1 < m_1 < j_2 < m_2 < \cdots$ then there exist $B \subseteq \mathbb{N}$ and an infinite set $M \subseteq \mathbb{N}$ with the properties: (i) $(m_{r-1}, m_r) \cap B = \{j_r\}$ for $r \in M$; (ii) $card([m_{r-1}, m_r] \cap B) \leq f(r)$ for each $r \in \mathbb{N} \setminus M$; (iii) $\left(\sum_{j\in B} x_{ij}\right)_i$ is a Cauchy sequence.*

*Then $(x_{ij})_i$ are Cauchy sequences uniformly on $j \in \mathbb{N}$.*

*Proof.* We first prove that $(x_{ij})_j$ converges to zero for each $i \in \mathbb{N}$. On the contrary, suppose that there exist $\varepsilon > 0$, $i_0 \in \mathbb{N}$ and a strictly increasing sequence of natural numbers $(j_r)_r$ with $j_r + 1 < j_{r+1}$ such that $\|x_{i_0 j_r}\| > \varepsilon$ for $r \in \mathbb{N}$.

Define $m_r = j_r + 1$ for $r \in \mathbb{N}$. Applying our hypothesis, the sequences $(j_r)_r$ and $(m_r)_r$ allow us to obtain two sets, $B$ and $M$, as in 2. From the former inequality it is obvious that, for each $r \in M$,

$$\left\| \sum_{\substack{j \in B \\ j \in (m_{r-1}, m_r)}} x_{i_0 j} \right\| = \|x_{i_0 j_r}\| > \varepsilon,$$

which contradicts that $\sum_{j \in B} x_{i_0 j}$ is a Cauchy series.

Having proved this preliminary step, we can now obtain our result. Suppose, contrary to our claim, that there exists $\varepsilon > 0$ such that for every $k \in \mathbb{N}$ we can choose $i \in \mathbb{N}$, $i > k$, and $j \in \mathbb{N}$ which verify the inequality $\|x_{ij} - x_{kj}\| > \varepsilon$. We now proceed by induction. The following argument, which gives us the first step, can then be applied to obtain the remaining inductive steps:

(i) Define $k_1 = 1$. By the previous assumption there exist $i_1 > k_1$ and $j_1$ such that $\|x_{i_1 j_1} - x_{k_1 j_1}\| > \varepsilon$.

(ii) Since $(x_{ij})_j$ converge to zero for each $i \in \mathbb{N}$, consider $m_1 \in \mathbb{N}$ which verifies the inequality $\|x_{ij}\| < \dfrac{\varepsilon}{7 \cdot 2^2 \cdot f(2)}$, for each $i \in \{1, 2, \dots, i_1\}$ and $j \geq m_1$. Let us observe that $m_1 > j_1$.

(iii) Since $(x_{ij})_i$ are Cauchy sequences, let $k_2$ be such that $\left\| \sum_{j \in C} (x_{ij} - x_{kj}) \right\| < \frac{\varepsilon}{7}$ for $i, k \geq k_2$ and $C \subseteq \{1, 2, \dots, m_1\}$. It is clear that $k_2 > i_1$.

In the second step, we apply this argument again, with the difference that $k_2$ has been already defined and we consider $\dfrac{\varepsilon}{7 \cdot 2^3 \cdot f(3)}$ to obtain $m_2$.

We continue in this fashion to complete this inductive argument and obtain four sequences $(k_r)_r$, $(i_r)_r$, $(j_r)_r$ and $(m_r)_r$ with $k_1 < i_1 < k_2 < i_2 < \dots$, $j_1 < m_1 < j_2 < m_2 < \dots$ such that, for $r > 1$,

(a) $\|x_{i_r j_r} - x_{k_r j_r}\| > \varepsilon$.

(b) $\left\| \sum_{j \in C} (x_{i_r j} - x_{k_r j}) \right\| < \frac{\varepsilon}{7}$ for each $C \subseteq \{1, 2, \dots, m_{r-1}\}$.

(c) $\|x_{ij}\| < \dfrac{\varepsilon}{7 \cdot 2^{r+1} \cdot f(r+1)}$ for each $i \in \{1, 2, \dots, i_r\}$ and $j \geq m_r$.

Let $B$ and $M$ be the sets that result from $(j_r)_r$ and $(m_r)_r$ in hypothesis 2. We are now in a position to obtain a contradiction. Combining (a) and (b), we have that for each $r \in M$, $r > 1$,

$$\left\|\sum_{j\in B}(x_{i_r j} - x_{k_r j})\right\| \geq \|x_{i_r j_r} - x_{k_r j_r}\| - \left\|\sum_{\substack{j\in B\\ j\leq m_{r-1}}}(x_{i_r j} - x_{k_r j})\right\| -$$

$$\left\|\sum_{\substack{j\in B\\ j\geq m_r}}(x_{i_r j} - x_{k_r j})\right\| > \varepsilon - \tfrac{\varepsilon}{7} - \sum_{k\geq r}\sum_{\substack{j\in B\\ j\in[m_k,m_{k+1})}}\|x_{i_r j}\| - \sum_{k\geq r}\sum_{\substack{j\in B\\ j\in[m_k,m_{k+1})}}\|x_{k_r j}\|.$$

From (c) we obtain, for $k \in \mathbb{N}$, $k \geq r$, $\sum_{\substack{j\in B\\ j\in[m_k,m_{k+1})}}\|x_{i_r j}\| < \frac{\varepsilon}{7\cdot 2^{k+1}\cdot f(k+1)}$. It is suffices to notice that $i_r \in \{1, 2, \dots i_k\}$ and $j \geq m_k$. The same inequality holds if $i_r$ is replaced by $k_r$, and thus

$$\left\|\sum_{j\in B}(x_{i_r j} - x_{k_r j})\right\| > \varepsilon - \tfrac{\varepsilon}{7} - 2\cdot\sum_{k\geq r}\frac{\varepsilon}{7\cdot 2^{k+1}\cdot f(k+1)} > \varepsilon - \tfrac{\varepsilon}{7} - \tfrac{2\cdot\varepsilon}{7}\cdot\sum_{k\geq r+1}\frac{1}{2^k} > \frac{4\cdot\varepsilon}{7},$$

which contradicts that $\left(\sum_{j\in B} x_{ij}\right)_i$ is a Cauchy sequence. ■

If $\mathcal{F}$ is a natural family with property $P_{c_0}$ the result above remains valid if hypotheses 1. and 2. are replaced by the condition: $\left(\sum_{j\in B} x_{ij}\right)_i$ is a Cauchy sequence for each $B \in \mathcal{F}$.

From Theorem 2.3 we next characterize the convergence of a sequence in $c_0(X)$, and so we prove the converse of the Basic Matrix Theorem for Banach spaces.

Corollary 2.4. *Let $X$ be a Banach space and let $(\overline{x}^i)_i$ be a sequence in $c_0(X)$, $\overline{x}^i = (x_{ij})_j$ for $i \in \mathbb{N}$. The following statements are equivalent:*

1. *There exists a natural family $\mathcal{F}$ with property SC such that $\left(\sum_{j\in B} x_{ij}\right)_i$ is a convergent sequence for each $B \in \mathcal{F}$.*
2. *There exists a natural family $\mathcal{F}$ with property $P_{c_0}$ such that $\left(\sum_{j\in B} x_{ij}\right)_i$ is a convergent sequence for each $B \in \mathcal{F}$.*
3. *The sequence $(\overline{x}^i)_i$ is convergent to some $\overline{x}^0 \in c_0(X)$.*

*Proof.* It is clear that we need only prove that 3.⇒1. Let $\mathcal{F}$ be the family of all subsets $B \subseteq \mathbb{N}$ such that $\left(\sum_{j\in B} x_{ij}\right)_i$ is convergent and denote $\overline{x}^0 = (x_j)_j$. It is obvious that $\phi_0(\mathbb{N}) \subseteq \mathcal{F}$.

First we claim that there exists a subsequence $(n_k)_k$ such that $\sum_k \|x_{in_k}\|$ converges for each $i \in \mathbb{N}$ or, equivalently, that there exists an infinite set $M \subseteq \mathbb{N}$ such that

$\sum_{j \in M} \|x_{ij}\|$ converges for each $i \in \mathbb{N}$.
Since $(x_{ij})_j$ converges to zero, it is well known that the above property holds for each sequence $(x_{ij})_j$ with $i \in \mathbb{N}$ fixed. Therefore, let us consider an infinite set $M_1 \subseteq \mathbb{N}$ with $\sum_{j \in M_1} \|x_{1j}\| < \infty$. The sequence $(x_{2j})_{j \in M_1}$ is also convergent to zero, so let $M_2 \subseteq M_1$ be an infinite set, with $\inf M_2 > \inf M_1$, such that $\sum_{j \in M_2} x_{2j}$ is absolutely convergent.

Completing this inductive argument, we obtain a sequence $(M_k)_k$ of infinite subsets of $\mathbb{N}$ with $M_{k+1} \subseteq M_k$, $\inf M_{k+1} > \inf M_k$ and $\sum_{j \in M_k} \|x_{kj}\| < \infty$ for $k \in \mathbb{N}$.

Define $M_0 = \{m_{11}, m_{21}, m_{31} \dots\}$, where $m_{k1}$ is the first element of $M_k$. It is easy to verify that $\sum_{j \in M_0} x_{ij}$ is absolutely convergent for $i \in \mathbb{N}$. Since $(x_j)_j$ is also convergent to zero, there exists an infinite set $M \subseteq M_0$ with $\sum_{j \in M} \|x_j\| < \infty$ and $\sum_{j \in M} \|x_{ij}\| < \infty$ for every $i \in \mathbb{N}$, which establishes the validity of our claim.

Next we prove the desired implication. In order to show that our family $\mathcal{F}$ has property SC, it is enough to construct an infinite set $P \subseteq \mathbb{N}$ such that $\lim_i \sum_{j \in P} x_{ij} = \sum_{j \in P} x_j$. We can inductively construct three strictly increasing sequences $(i_r)_r$, $(m_r)_r$ and $(j_r)_r$ of natural numbers such that, for $r > 1$,

(a) $\|\overline{x}^i - \overline{x}^0\| < \dfrac{1}{(r+1)2^{r+1}}$ for each $i \geq i_r$.

(b) $\sum_{j \in B \cap M} \|x_{ij}\| < \dfrac{1}{2^{r+1}}$ and $\sum_{j \in B \cap M} \|x_j\| < \dfrac{1}{2^{r+1}}$ for each $i \in \{1, 2, \dots, i_r\}$ and $B \subseteq \{m_r, \dots\}$.

(c) $m_r \leq j_r \leq m_{r+1}$ and $j_r \in M$ for $r \in \mathbb{N}$.

Let $P = \{j_r : r \in \mathbb{N}\}$. We next prove that $\lim_i \sum_{j \in P} x_{ij} = \sum_{j \in P} x_j$.
Consider $\varepsilon > 0$ and $r \in \mathbb{N}$ with $\frac{3}{2^{r+1}} < \varepsilon$. Combining (a) and (b) we obtain, for each $i \in \mathbb{N}$, $i \geq i_r$,

$$\left\| \sum_{j \in P} x_{ij} - \sum_{j \in P} x_j \right\| \leq \left\| \sum_{\substack{j \in P \\ j \leq j_{r-1}}} (x_{ij} - x_j) \right\| + \sum_{\substack{j \in P \\ j \geq m_r}} \|x_{ij}\| - \sum_{\substack{j \in P \\ j \geq m_r}} \|x_j\| < \frac{3}{2^{r+1}} < \varepsilon. \qquad \blacksquare$$

*Remark* 2.5 Fleischer [6] proved the following result: Let $(x_{ij})_{i,j}$ be a matrix from a quasi-normed group whose columns are Cauchy and every infinite subset of indices has, for every $\varepsilon > 0$, an infinite subset $J$ for which

$$\limsup_{i,i' \in J} \limsup_{\text{finite } F \uparrow J} \sup_{\text{finite } F' \subseteq J} \left| \sum_{j \in F' \setminus F} x_{ij} - x_{i'j} \right| < \varepsilon.$$

Then the columns are uniformly Cauchy.

Let $(x_{ij})_{ij}$ be a matrix in a Banach space $X$ whose rows are convergent to zero. Under the assumptions of Fleischer's result above, there exists $\overline{x}^0 = (x_j)_j$ such that $\lim_i x_{ij} = x_j$ uniformly on $j \in \mathbb{N}$. Define $\overline{x}^i = (x_{ij})_j \in c_0(X)$ for each $i \in \mathbb{N}$, it follows that $(\overline{x}^i)_i$ converges to $\overline{x}^0$ in $c_0(X)$. If we consider a matrix $(x_{ij})_{i,j}$ in $X$ which does not verify Fleischer's assumptions, then it is easy to check that $(\overline{x}^i)_i$ is not a Cauchy sequence in $c_0(X)$. Thus it can be seen that properties 1., 2. and 3. in Corollary 2.4 are equivalent to the following: 4. The matrix $(x_{ij})_{i,j}$ verifies Fleischer's assumptions.

We are interested in matrix results which are based on separation and supremum properties of natural families. Let us observe that the natural families we consider in this paper have established their own properties previously. Hence, Fleischer's techniques are not strongly connected with ours. ■

In the following remark we give some examples of natural families with property $P_{c_0}$ which are not $SC$.

*Remark* 2.6 We now introduce a supremum property for natural families which implies $P_{c_0}$.

We say that a natural family $\mathcal{F}$ has property $P_{c_0}^{\sigma}$ if for every pair $(j_r)_r$, $(m_r)_r$ of sequences in $\mathbb{N}$ with $j_1 < m_1 < j_2 < m_2 < \dots$ there exist an infinite set $M \subseteq \mathbb{N}$, $M' \subseteq M$, $B \in \mathcal{F}$ and $\{l_r : r \in M'\} \subseteq \mathbb{N}$ such that $B = \{j_r : r \in M\} \cup \{l_r : r \in M'\}$ and, for each $r \in M'$ and $h \in M$ with $r < h$, $m_r \leq l_r \leq m_{h-1}$.

If $\mathcal{F}$ denotes a natural family with property $P_{c_0}^{\sigma}$, let us prove that $\mathcal{F}$ has also property $P_{c_0}$. Let $f : \mathbb{N} \to \mathbb{N}$ be the map given by $f(i) = 1$ for $i \in \mathbb{N}$. For every pair of sequences $(j_r)_r$ and $(m_r)_r$ in $\mathbb{N}$ with $j_1 < m_1 < j_2 < m_2 < \dots$, let $M \subseteq \mathbb{N}$, $M' \subseteq M$, $B \in \mathcal{F}$ and $\{l_r : r \in M'\}$ be the sets which verify the properties that appear in the definition of $P_{c_0}^{\sigma}$.
We prove that $M$ and $B$ satisfy properties $(a)$ and $(b)$ in Definition 2.2. It suffices to show that:

(i) $(m_{r-1}, m_r) \cap \{l_s : s \in M'\}$ is empty if $r \in M$

(ii) $[m_{r-1}, m_r] \cap \{l_s : s \in M'\}$ is empty or a singleton if $r \in \mathbb{N} \setminus M$.

First, in order to prove (i), let us consider $r \in M$ and $k \in M'$ such that $l_k \in (m_{r-1}, m_r)$ and $m_k \leq l_k$ (according to the above definition). We distinguish three cases:

(a.1) If $k > r$, then we have $m_r < m_k \leq l_k$, which contradicts the previous assumption.

(b.1) If $k = r$, then we have $m_r = m_k \leq l_k \leq m_r$ and so $l_k = m_r$, which is impossible.

(c.1) If $k < r$ it follows that $m_k \leq l_k \leq m_{r-1}$ and so $l_k = m_{r-1}$, which is also impossible.

We now prove (ii). Let $k \in M'$ and $r \in \mathbb{N} \setminus M$ be such that $l_k \in [m_{r-1}, m_r]$ and $k$ is the largest possible in $M'$ satisfying the previous property. Consider $k_1 = \inf\{k' \in M : k' > k\}$. Then we have $m_k \leq l_k \leq m_{k_1-1}$, and consequently, $m_{r-1} \leq l_k \leq m_{k_1-1}$, $m_k \leq l_k \leq m_r$. From this, we obtain $k < r < k_1$. If there exists $l_{k'} \in [m_{r-1}, m_r]$ with $k' < k$, we will have $m_{k'} \leq l_{k'} \leq m_{k-1}$, which contradicts the inequality $k < r$. This completes the proof.

The following example shows that there exist natural families with property $P_{c_0}$ that lack property $SC$. Let $\mathcal{B}_1$ be the family of sets $B \subseteq \mathbb{N}$ with the following properties:

(a) $B$ and $B^c$ contain infinitely many even numbers and odd numbers.

(b) $\{n \in \mathbb{N} : \{4n-1, 4n\} \subseteq B\}$ is finite.

Let $\mathcal{F}_1 = \mathcal{B}_1 \cup \phi_0(\mathbb{N})$. In order to prove that $\mathcal{F}_1$ has the property $P_{c_0}$, let us consider the map $f : \mathbb{N} \to \mathbb{N}$ given by $f(i) = 1$ for $i \in \mathbb{N}$ and let $(j_r)_r$ and $(m_r)_r$ be two sequences in $\mathbb{N}$ with $j_1 < m_1 < j_2 < m_2 < \dots$. Define $A = \bigcup_r \{j_r\}$, $B_1 = \{2n : n \in \mathbb{N}\} - \{4n : n \in \mathbb{N}\}$, $B_2 = \{4n : n \in \mathbb{N}\}$, $C_1 = \{2n-1 : n \in \mathbb{N}\} - \{4n-1 : n \in \mathbb{N}\}$ and $C_2 = \{4n-1 : n \in \mathbb{N}\}$. In order to construct the infinite set $M \subseteq \mathbb{N}$ and $B \in \mathcal{F}_1$ which verify $(a)$ and $(b)$ in Definition 2.2, we distinguish between two cases and proceed by induction in each of them:

(A) $A \cap \{2n : n \in \mathbb{N}\}$ is infinite.
In the first step of our inductive argument, we consider:

(i.1) $r_{11} \in \mathbb{N}$ such that $j_{r_{11}}$ is an even number.

(ii.1) $r'_{11} \in \mathbb{N}$, $r'_{11} > r_{11}$ satisfying $[m_{r'_{11}-1}, m_{r'_{11}}) \cap (B_1 \cup B_2) \neq \emptyset$. Let $l_{r'_{11}}$ be an even number which belongs to the previous intersection set.

(iii.1) $r_{12} \in \mathbb{N}$, $r_{12} > r'_{11}$, which verifies $[m_{r_{12}-1}, m_{r_{12}}) \cap C_1 \neq \emptyset$. Let $k_{r_{12}}$ be an element which belongs to this intersection set.

Define $A^1 = \{j_{r_{11}}, k_{r_{12}}\}$ and $M^1 = \{r_{11}\}$, let us observe that $j_{r_{11}} < m_{r_{11}} \leq m_{r'_{11}-1} \leq l_{r'_{11}} < m_{r'_{11}} \leq m_{r_{12}-1} \leq k_{r_{12}} < m_{r_{12}}$.

In the second step we apply a similar argument. Let us consider:

(i.2) $r_{21} \in \mathbb{N}$, $r_{21} > r_{12}$ such that $j_{r_{21}}$is an even number.

(ii.2) $r'_{21} \in \mathbb{N}$, $r'_{21} > r_{21}$ verifying $[m_{r'_{21}-1}, m_{r'_{21}}) \cap (B_1 \cup B_2) \neq \emptyset$, so we can consider $l_{r'_{21}}$ which belongs to this intersection set.

(iii.2) $r_{22} \in \mathbb{N}$, $r_{22} > r'_{21}$, such that there exists $k_{r_{22}} \in \mathbb{N}$ with $k_{r_{22}} \in [m_{r_{22}-1}, m_{r_{22}}) \cap C_1$.

Define $A^2 = \{j_{r_{21}}, k_{r_{22}}\}$ and $M^2 = \{r_{21}\}$, it is clear that $m_{r_{12}} \leq m_{r_{21}-1} < j_{r_{21}} < m_{r_{21}} \leq m_{r'_{21}-1} \leq l_{r'_{21}} < m_{r'_{21}} \leq m_{r_{22}-1} \leq k_{r_{22}} < m_{r_{22}}$.

We continue in this fashion to complete this inductive argument and so obtain three strictly increasing sequences: $(j_{r_{i1}})_i$, $(l_{r'_{i1}})_i$, two sequences of even numbers, and $(k_{r_{i2}})_i$, whose elements belong to $C_1$, with $j_{r_{i1}} < m_{r_{i1}} \leq m_{r'_{i1}-1} \leq l_{r'_{i1}} < m_{r'_{i1}} \leq m_{r_{i2}-1} \leq k_{r_{i2}} < m_{r_{i2}} \leq m_{r_{(i+1)1}-1} < j_{r_{(i+1)1}}$ for $i \in \mathbb{N}$. Let us define $A^n = \{j_{r_{n1}}, k_{r_{n2}}\}$, $M^n = \{r_{n1}\}$, $B = \bigcup_n A^n$ and $M = \bigcup_n M^n$. It is obvious that $B$ and $M$ verify properties $(a)$ and $(b)$ in Definition 2.2 and also $B \in \mathcal{F}_1$.

(B) $A \cap \{2n-1 : n \in \mathbb{N}\}$ is infinite.
This can be proved by using an argument similar to that in the previous case, with the following differences: for each $i \in \mathbb{N}$ $j_{r_{i1}}$ and $l_{r'_{i1}}$ must be odd numbers and $(k_{r_{i2}})_i$ must be a sequence in $B_1$. This completes the proof

To see that $\mathcal{F}_1$ does not have property SC, it is sufficient to observe that there is no infinite subset of the set of all multiples of 4 that belongs to $\mathcal{F}_1$.

As in the analysis of the previous family, it can be shown that the following families have property $P_{c_0}$ and lack property SC. It is enough to consider an inductive argument easier than the previous one. The same notation used in the above argument will be followed. Let $Q_1$ be an infinite subset of $\mathbb{N}$ whose complementary set is also infinite. Let $Q_2$ and $Q_3$ be two infinite subsets of $\mathbb{N}$ such that at least one of them has an infinite complement. We consider and study the following families:

- $\mathcal{F}_\alpha = \{B \subseteq \mathbb{N} : B \cap Q_1 \text{ is infinite}\} \cup \phi_0(\mathbb{N})$. We can distinguish between two cases: $(i)$ if $A \cap Q_1$ is infinite (according to the previous notation, we have $A = \bigcup_r \{j_r\}$), an easier inductive argument allows us to construct the sequence $(j_{r_{i1}})_i$ in $Q_1$. We need only consider $(i.1), (i.2), \ldots$; $(ii)$ if $A \cap Q_1$ is a finite set, we need only construct the sequences $(j_{r_{i1}})_i$ in $A$ and $(k_{r_{i2}})_i$ in $Q_1$.
The complement of $Q_1$ allows us to show that $\mathcal{F}_\alpha$ does not have property SC.

- $\mathcal{F}_\beta = \{B \subseteq \mathbb{N} : B \cap Q_1 \text{ and } B^c \cap Q_1 \text{ are infinite}\} \cup \phi_0(\mathbb{N})$. If $A \cap Q_1$ is infinite, it is enough to construct the sequences $(j_{r_{i1}})_i$ and $(l_{r'_{i1}})_i$ whose elements belong to $Q_1$. If $A \cap Q_1$ is a finite set, we can construct $(j_{r_{i1}})_i$ in $A$ and the sequences $(l_{r'_{i1}})_i$, $(k_{r_{i2}})_i$ in $Q_1$.
Then we can observe that there is no infinite subset of $Q_1^c$ which belongs to $\mathcal{F}_\beta$ and so this family does not have property SC.

- $\mathcal{F}_\sigma = \{B \subseteq \mathbb{N} : B \cap Q_2 \text{ and } B \cap Q_3 \text{ are infinite}\} \cup \phi_0(\mathbb{N})$. If either $A \cap Q_2$ or $A \cap Q_3$ is infinite, the sequences $(j_{r_{i1}})_i$ and $(k_{r_{i2}})_i$ can be constructed so that $j_{r_{i1}} \in Q_2$ or $Q_3$, respectively, and $k_{r_{i2}} \in Q_3$ or $Q_2$, respectively, for $i \in \mathbb{N}$. If $A \cap Q_2$ and $A \cap Q_3$ are both finite sets, a similar argument allows us to obtain two sequences: $(j_{r_{i1}})_i$ in $A$ and $(k_{r_{i2}})_i$ whose even terms belong to $Q_2$ and whose odd terms belong to $Q_3$.
$Q_2^c$ and $Q_3^c$ are not both finite sets and therefore $\mathcal{F}_\sigma$ lacks property SC.

- $\mathcal{F}_\theta = \{B \subseteq \mathbb{N} : B \cap Q_2, B \cap Q_3, B^c \cap Q_2$ and $B^c \cap Q_3$ are infinite$\} \cup \phi_0(\mathbb{N})$. If either $A \cap Q_2$ or $A \cap Q_3$ is infinite, the sequences $(j_{r_{i1}})_i$ and $(k_{r_{i2}})_i$ can be constructed so that $j_{r_{i1}} \in Q_2$ or $Q_3$, respectively, and $k_{r_{i2}} \in Q_3$ or $Q_2$, respectively, for $i \in \mathbb{N}$. Also, we must obtain the sequence $(l_{r'_{i1}})_i$ whose even terms belong to $Q_2$ and whose odd terms belong to $Q_3$. If $A \cap Q_2$ and $A \cap Q_3$ are both finite sets, we can proceed analogously to the above case with the difference that $j_{r_{i1}} \in A$ for $i \in \mathbb{N}$ and the sequence $(k_{r_{i2}})_i$ verifies $k_{r_{i2}} \in Q_2$ if $i$ is an even number and $k_{r_{i2}} \in Q_3$ if $i$ is an odd number.
  It is clear that $\mathcal{F}_\theta$ does not have property SC.

■

## 3 Convergence in $cs(X)$

Let $X$ be a Banach space and let $cs(X)$ be the space

$$\left\{\overline{x} = (x_j)_j : \sum_j x_j \text{ is convergent}\right\},$$

endowed with the norm

$$\|(x_i)_i\| = \sup_n \left\{\left\|\sum_{i=1}^n x_i\right\|\right\}.$$

It is clear that $cs(X)$ is complete and that a sequence $(\overline{x}^i)_i$, with $\overline{x}^i = (x_{ij})_j$ for each $i \in \mathbb{N}$, converges to $\overline{x}^0 = (x_j)_j$ in $cs(X)$ if and only if $\lim_i \sum_{j\in F} x_{ij} = \sum_{j\in F} x_j$ uniformly on the family $I_0(\mathbb{N})$ of the finite intervals $F$ in $\mathbb{N}$.

Analysis similar to that in the previous section allows us to characterize the convergence of a sequence $(\overline{x}^i)_i$ in $cs(X)$. The result of Swartz's [12] we mentioned in the introduction can be improved in order to obtain a sufficient condition for the above convergence. This matrix result, which we will study later, is based on the following supremum property ([9]): A natural family $\mathcal{F}$ is called an *IQ* $\sigma$-family if, for every sequence $(F_i)_i$ of intervals in $\mathbb{N}$ such that $\sup F_i < \inf F_{i+1}$, for each $i \in \mathbb{N}$, there exists an infinite set $M \subseteq \mathbb{N}$ such that $B = \bigcup_{i\in M} F_i \in \mathcal{F}$.
In order to establish not only the abovementioned improvement, but also better matrix results and a necessary condition for the convergence of a sequence in $cs(X)$, we consider the following separation properties of natural families:

Definition 3.1. *Let $\mathcal{F}$ be a natural family. We say that $\mathcal{F}$ has property $P_0$ if for every sequence $(F_i)_i$ in $I_0(\mathbb{N})$ with $\sup F_i < \inf F_{i+1}$, for each $i \in \mathbb{N}$, there exists an infinite set $M \subseteq \mathbb{N}$ and $B \in \mathcal{F}$ satisfying $F_i \subseteq B$ for each $i \in M$.*

The following result can be easily verified:

Lemma 3.2. *Let $\mathcal{F}$ be a natural family with property $P_0$ and let $\sum_i x_i$ be a $\mathcal{F}$-Cauchy series in the normed space $X$. Then $\sum_i x_i$ is a Cauchy series.*

DEFINITION 3.3. *We say that a natural family $\mathcal{F}$ has property $P_1$ if there exists a map $f : \mathbb{N} \to \mathbb{N}$ such that for every pair $(F_r)_r$, $(m_r)_r$ of sequences in $I_0(\mathbb{N})$ and $\mathbb{N}$, respectively, with $m_r \leq \inf F_r \leq \sup F_r < m_{r+1}$ for each $r \in \mathbb{N}$, there exist $B \in \mathcal{F}$ and an infinite set $M \subseteq \mathbb{N}$ such that:*

*(a) For $r \in M$, $B \cap [m_r, m_{r+1}) = F_r$.*

*(b) For $r \in \mathbb{N} \setminus M$ and $r > 1$, $B \cap (m_r, m_{r+1})$ is either empty or can be written as the union of at most $f(r-1)$ intervals.*

It is obvious that each natural family with property $P_1$ has property $P_0$, and also that each $IQ$ $\sigma$-family has property $P_1$. However, it will be shown in the last remark of this section that these two properties are not equivalent.

LEMMA 3.4. *Assume that $\mathcal{F}$ is a natural family with property $P_1$ and $(x_{ij})_{i,j}$ is a matrix in the Banach space $X$ such that $\left(\sum_{j\in B} x_{ij}\right)_i$ is a convergent sequence for each $B \in \mathcal{F}$. Then $\left(\sum_j x_{ij}\right)_i$ is convergent.*

*Proof.* On the contrary, consider $\varepsilon > 0$ such that for every $k \in \mathbb{N}$ there exists $i > k$ with $\left\|\sum_j x_{ij} - \sum_j x_{kj}\right\| > \varepsilon$. Since $\mathcal{F}$ has property $P_1$, we can choose a map $f$ as in Definition 3.3. The proof is based on the following inductive argument:
We establish the first inductive step, to which the following steps are very similar:

(i) Consider $k_1 = 1$, the above assumption allows us to consider $i_1 > k_1$ with $\left\|\sum_j (x_{i_1 j} - x_{k_1 j})\right\| > \varepsilon$ .

(ii) From Lemma 3.2, let $m_1 \in \mathbb{N}$ be such that $\left\|\sum_{j\in F} x_{ij}\right\| < \frac{\varepsilon}{7\cdot 2^2\cdot f(1)}$, for $F \in I_0(\mathbb{N})$, $\inf F \geq m_1$ and $i \in \{1, 2, \ldots, i_1\}$, and so $\left\|\sum_{j\geq m_1} x_{ij}\right\| \leq \frac{\varepsilon}{7\cdot 2^2\cdot f(1)}$. Define $F_1 = [1, m_1)$, then by (i) and the previous inequality we have $\left\|\sum_{j\in F_1} (x_{i_1 j} - x_{k_1 j})\right\| > \dfrac{6\varepsilon}{7}$.

(iii) Since $(x_{ij})_i$ is a Cauchy sequence, there exists $k_2 \in \mathbb{N}$, $k_2 > i_1$, satisfying $\left\|\sum_{j\in C} (x_{pj} - x_{qj})\right\| < \frac{\varepsilon}{7}$, for $p, q \geq k_2$ and $C \subseteq \{1, 2, \ldots, m_1\}$.

In the second step, we apply this argument again, except that $k_2$ has already been defined, we consider $\dfrac{\varepsilon}{7\cdot 2^3\cdot f(2)}$ to obtain $m_2$ (from (iii) we have $m_2 > m_1$) and $F_2$ denotes the interval $[m_1, m_2)$.

It is now easy to complete this inductive argument and to obtain three strictly increasing sequences of natural numbers $(k_r)_r$, $(i_r)_r$ and $(m_r)_r$ with $k_1 < i_1 < k_2 < i_2 < \dots$ such that, for $r > 1$, the following properties hold:

(a) $\left\| \sum_{j \in F_r} (x_{i_r j} - x_{k_r j}) \right\| > \frac{5\varepsilon}{7}$, where $F_r$ denotes the interval in $\mathbb{N}$ $[m_{r-1}, m_r)$.

(b) $\left\| \sum_{j \in C} (x_{pj} - x_{qj}) \right\| < \frac{\varepsilon}{7}$, for $C \subseteq \{1, 2, \dots, m_{r-1}\}$ and $p, q \geq k_r$.

(c) $\left\| \sum_{j \in F} x_{ij} \right\| < \frac{\varepsilon}{7 \cdot 2^{r+1} \cdot f(r)}$, for $i \in \{1, 2, \dots, i_r\}$ and $F \in I_0(\mathbb{N})$, $\inf F \geq m_r$.

For the sequences $(F_r)_{r>1}$ and $(m_{r-1})_{r>1}$, there exist $B \in \mathcal{F}$ and an infinite set $M \subseteq \mathbb{N}$ with the properties (a) and (b) in Definition 3.3. An analysis similar to that at the end of the proof of Theorem 2.3 shows that $\left( \sum_{j \in B} x_{ij} \right)_i$ cannot be a Cauchy sequence, which contradicts our hypothesis. ■

Lemma 3.5. *Let $\mathcal{F}$ be a natural family with property $P_1$ and let $(x_{ij})_{i,j}$ be a matrix in the Banach space $X$ such that the sequence $\left( \sum_{j \in B} x_{ij} \right)_i$ is convergent for each $B \in \mathcal{F}$. Then $\left( \sum_{j \in A_m} x_{ij} \right)_i$ are uniformly convergent on $m \in \mathbb{N}$, where $A_m = (m, +\infty) \cap \mathbb{N}$.*

*Proof.* The proof is based on an inductive argument similar to that in the proof of Lemma 3.4, so we only sketch our argument.

If the result is false, consider $\varepsilon > 0$ such that for every $k, m \in \mathbb{N}$ there exist $i \in \mathbb{N}$, $i > k$, and $h \in \mathbb{N}$, $h + 1 > m$, with $\left\| \sum_{j > h} (x_{ij} - x_{kj}) \right\| > \varepsilon$. Lemma 3.4 gives that $\left( \sum_j x_{ij} \right)_i$ and $\left( \sum_{j > m} x_{ij} \right)_i$, for $m \in \mathbb{N}$, are Cauchy sequences.

Since $\mathcal{F}$ has property $P_1$, we can choose a map $f$ as in Definition 3.3. The following argument, which gives us the first step, is very similar to that in the remaining inductive steps:

(i) Choose $k_1 \in \mathbb{N}$ such that $\left\| \sum_j (x_{ij} - x_{kj}) \right\| < \frac{\varepsilon}{8}$, for $i, k \geq k_1$.

(ii) Consider $m_1 = 1$. By the previous assumption there exist $i_1 > k_1$ and $h_1 > m_1$, with $\left\| \sum_{j \geq h_1} (x_{i_1 j} - x_{k_1 j}) \right\| > \varepsilon$. Define $F_1 = [m_1, h_1)$, by (i) and the previous inequality it follows that

$$\left\|\sum_{j\in F_1}(x_{i_1j}-x_{k_1j})\right\| = \left\|\sum_j(x_{i_1j}-x_{k_1j}) - \sum_{j\ge h_1}(x_{i_1j}-x_{k_1j})\right\| \ge$$
$$\left\|\sum_{j\ge h_1}(x_{i_1j}-x_{k_1j})\right\| - \left\|\sum_j(x_{i_1j}-x_{k_1j})\right\| > \tfrac{7\cdot\varepsilon}{8}.$$

(iii) Consider $m_2 \in \mathbb{N}$, $m_2 > h_1$, such that $\left\|\sum_{j\in F} x_{ij}\right\| < \frac{\varepsilon}{8\cdot 2^3\cdot f(1)}$, for $F \in I_0(\mathbb{N})$, $\inf F \ge m_2$ and $i \in \{1, 2, \ldots, i_1\}$.

In the second step we apply this argument again, except that $m_2$ has already been defined, (i) uses that $\left(\sum_{j\ge m_2} x_{ij}\right)_i$ is a Cauchy sequence, $k_2$ is chosen satisfying $k_2 > i_1$ and we consider $\dfrac{\varepsilon}{8\cdot 2^4\cdot f(2)}$ to obtain $m_3$.

We continue in this fashion to complete this inductive argument. As in the proof of Lemma 3.4, we can now consider the sequences $(F_r)_{r\ge1}$, $(m_r)_{r\ge1}$ and analyse our hypothesis in order to obtain a contradiction which proves the lemma. ■

Theorem 3.6. *Assume that $\mathcal{F}$ is a natural family with property $P_1$ and $(x_{ij})_{i,j}$ is a matrix in the Banach space $X$ such that $\left(\sum_{j\in B} x_{ij}\right)_i$ is a convergent sequence for each $B \in \mathcal{F}$. If $x_j$ denotes the limit of the sequence $(x_{ij})_i$, for $j \in \mathbb{N}$, then we have*

1. $\lim_i \left(\sum_j x_{ij}\right) = \sum_j x_j$.

2. $\lim_i \sum_{j\in F} x_{ij} = \sum_{j\in F} x_j$ *uniformly on* $F \in I_0(\mathbb{N})$.

3. *If for every $B \in \mathcal{F}$ the series $\sum_i \left(\sum_{j\in B} x_{ij}\right)$ is convergent, then the series $\sum_i\sum_j x_{ij}$ and $\sum_j\sum_i x_{ij}$ converge and their sums are equal.*

*Proof.* From Lemmas 3.4 and 3.5 and the inequality

$$\left\|\sum_{j=1}^{m}(x_{ij}-x_{kj})\right\| \le \left\|\sum_j(x_{ij}-x_{kj}) - \sum_{j>m}(x_{ij}-x_{kj})\right\| \le$$
$$\left\|\sum_j(x_{ij}-x_{kj})\right\| + \left\|\sum_{j>m}(x_{ij}-x_{kj})\right\|$$

it follows that $\left(\sum_{j=1}^{m} x_{ij}\right)_i$ are uniformly convergent on $m \in \mathbb{N}$. Let $\alpha$ denote the limit of the sequence $\left(\sum_j x_{ij}\right)_i$. For every $\varepsilon > 0$ there exist $i_0, m_0 \in \mathbb{N}$ verifying:

(i) $\left\|\sum_j x_{ij} - \alpha\right\| < \frac{\varepsilon}{3}$, for $i \geq i_0$.

(ii) $\left\|\sum_{j=1}^{m}(x_{ij} - x_j)\right\| < \frac{\varepsilon}{3}$, for $i \geq i_0$ and $m \in \mathbb{N}$.

(iii) $\left\|\sum_{j=1}^{m} x_{i_0 j} - \sum_j x_{i_0 j}\right\| < \frac{\varepsilon}{3}$, for $m \geq m_0$.

From this it is obvious that $\left\|\sum_{j=1}^{m} x_j - \alpha\right\| < \varepsilon$ for each $m \geq m_0$, and so $\alpha = \sum_j x_j$.

It is easily seen that $\lim_i \left(\sum_{j\in F} x_{ij}\right) = \sum_{j\in F} x_j$, $\lim_i \left(\sum_{j\notin F} x_{ij}\right) = \sum_{j\notin F} x_j$ uniformly on $F \in I_0(\mathbb{N})$ and $\lim_i \left(\sum_{j\geq m} x_{ij}\right) = \sum_{j\geq m} x_j$ uniformly on $m \in \mathbb{N}$.

Let us prove 3. If we let $z_{lj} = \sum_{i\leq l} x_{ij}$, then the matrix $(z_{lj})_{l,j}$ verifies the hypothesis of this theorem and therefore

$$\sum_i \sum_j x_{ij} = \lim_l \sum_j z_{lj} = \sum_j \sum_i x_{ij}.$$

This completes the proof. ■

*Remark* 3.7

a) Let $(x_{ij})_{i,j}$ be a matrix in the Banach space $X$ such that $\sum_i \sum_j x_{ij} = \sum_j \sum_i x_{ij}$. As a consequence of Moore's lemma ([5]), it can be deduced that the net $\left(\sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij}\right)_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ converges if we consider, on $\mathbb{N} \times \mathbb{N}$, the order relation: $(n, m) \leq (n', m')$ if and only if $n \leq n'$ and $m \leq m'$. We also have that

$$\lim_{n,m} \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} = \sum_i \sum_j x_{ij} = \sum_j \sum_i x_{ij}$$

b) Let $(x_{ij})_{i,j}$ be a matrix in the Banach space $X$ such that:

(1) $\sum_i x_{ij}$ converges for each $j \in \mathbb{N}$.

(2) $\sum_j x_{ij}$ converges for each $i \in \mathbb{N}$.

(3) For every sequence $(F_r)_r$ of intervals in $\mathbb{N}$, $F_r = [p_r, q_r]$ with $p_r < q_r < p_{r+1}$ for each $r \in \mathbb{N}$, there exists a subsequence $(F_{r_k})_k$ such that $\sum_i \sum_k \sum_{j\in F_{r_k}} x_{ij}$ converges.

Then, Li Ronglu and Shin Min Kang [10] proved that the series $\sum_{i,j} x_{ij}$, $\sum_i \sum_j x_{ij}$ and $\sum_j \sum_i x_{ij}$ converge and their sums are equal. Proceeding as in the proofs of Lemma 3.4, Lemma 3.5 and Theorem 3.6, it is easily shown that the result above remains valid if (3) is replaced by the condition: there exists a map $f : \mathbb{N} \to \mathbb{N}$ such that for every pair $(F_r)_r$, $(m_r)_r$ of sequences in $I_0(\mathbb{N})$ and $\mathbb{N}$, respectively, with $m_r \leq \inf F_r \leq \sup F_r < m_{r+1}$ for each $r \in \mathbb{N}$, there exist $B \in \mathcal{F}$ and an infinite set $M \subseteq \mathbb{N}$ as in Definition 3.3 such that the series $\sum_i \sum_{j \in B} x_{ij}$ converges. Hence, for Banach spaces, we obtain a generalization of the aforementioned result.

c) We have already referred to the matrix result obtained by Swartz [12], which considers an $IQ$ $\sigma$-family and generalizes Antosik interchange theorem ([2]). For Banach spaces, this result is similar to our Theorem 3.6 (without 2.), but we consider a family with property $P_1$ instead of an $IQ$ $\sigma$-family.

■

Theorem 3.6 enables us to give a characterization of the convergence in the space $cs(X)$.

COROLLARY 3.8. *Let $X$ be a Banach space and let $(\overline{x}^i)_i$ be a sequence in $cs(X)$. The following statements are equivalent:*

1. *There exists an $IQ$ $\sigma$-family $\mathcal{F}$ such that $\left(\sum_{j \in B} x_{ij}\right)_i$ is convergent, for each $B \in \mathcal{F}$.*

2. *There exists a natural family $\mathcal{F}$ with property $P_1$ such that $\left(\sum_{j \in B} x_{ij}\right)_i$ is convergent, for each $B \in \mathcal{F}$.*

3. *The sequence $(\overline{x}^i)_i$ is convergent to some $\overline{x}^0 = (x_j)_j \in cs(X)$.*

*Proof.* It is sufficient to prove that 1. is a consequence of 3. Write $\overline{x}^i = (x_{ij})_j$ for $i \in \mathbb{N}$. In order to construct the $IQ$ $\sigma$-family $\mathcal{F}$ define $\mathcal{F} = \left\{ B \subseteq \mathbb{N} : \left(\sum_{j \in B} x_{ij}\right)_i \text{ is convergent} \right\}$ and let us consider a sequence $(F_i)_i$ in $I_0(\mathbb{N})$ with $\sup F_i < \inf F_{i+1}$ for each $i \in \mathbb{N}$. It is enough to show that there exists a subsequence $(F_{i_k})_k$ such that $\left(\sum_{j \in B} x_{ij}\right)_i$ is convergent for $B = \bigcup_k F_{i_k}$.

Let $y_{il} = \sum_{j \in F_l} x_{ij}$ and $y_l = \sum_{j \in F_l} x_j$ for $l \in \mathbb{N}$. The sequences $\overline{y}^i = (y_{il})_l$ and $\overline{y}^0 = (y_l)_l$ belong to $c_0(X)$ and, in this space, $\lim_i \overline{y}^i = \overline{y}^0$. As in the proof of Corollary 2.4, we can construct an infinite set $M \subseteq \mathbb{N}$ such that $\lim_i \sum_{l \in M} y_{il} = \sum_{l \in M} y_l$. Let $B = \bigcup_{l \in M} F_l$. It is obvious that $\sum_{l \in M} y_{il} = \sum_{j \in B} x_{ij}$ and $\sum_{l \in M} y_l = \sum_{j \in B} x_j$. Therefore $\lim_i \sum_{j \in B} x_{ij} = \sum_{j \in B} x_j$.

■

*Remark* 3.9 Let $X$ be a Banach space. Corollary 2.4 and the isomorphism between $cs(X)$ and $c_0(X)$ allows us to obtain another characterization of the convergence of a sequence in $cs(X)$. We will first describe the isomorphism which will be considered in this argument.
Let $\phi : cs(X) \to c(X)$ be given by $\phi((x_1, x_2, \dots, x_n, \dots)) = (x_1, x_1 + x_2, \dots, \sum_{k=1}^n x_k, \dots)$ for $\overline{x} = (x_i)_i \in cs(X)$. For an element $\overline{y} = (y_i)_i \in c(X)$ we can consider the sequence $(y_1, y_2 - y_1, \dots, y_n - y_{n-1}, \dots) = \phi^{-1}(\overline{y})$ which belongs to $cs(X)$, and so it is enough to observe that $\|\phi(\overline{x})\| = \|\overline{x}\|$ to conclude that $c(X)$ and $cs(X)$ are linearly isometric.
Let $\varphi$ be the isomorphism from $c(X)$ to $c_0(X)$ given by $\varphi((y_i)_i) = (y_0, y_1 - y_0, \dots, y_{n-1} - y_0, \dots)$ for $\overline{y} = (y_i)_i \in c(X)$ and $y_0 = \lim_i y_i$. Also, $\varphi^{-1}((z_i)_i) = (z_2 + z_1, z_3 + z_1, \dots, z_{n+1} + z_1, \dots)$ for each $\overline{z} = (z_i)_i \in c_0(X)$.

If $(\overline{x}^i)_i$ denotes a sequence in $cs(X)$ (or $c(X)$ or $c_0(X)$) we will write $\overline{x}^i = (x_{ij})_j$ for $i \in \mathbb{N}$. Using the above notation, we obtain the following characterization:

*Let $X$ be a Banach space and let $(\overline{x}^i)_i$ be a sequence in $cs(X)$. The following conditions are equivalent:*

1. *There exists a natural family $\mathcal{F}$ with property SC such that* $\left(\sum_{k\in B}\sum_{j\geq k} x_{ij}\right)_i$ *is a convergent sequence for each $B \in \mathcal{F}$.*
2. *There exists a natural family $\mathcal{F}$ with property $P_{c_0}$ such that* $\left(\sum_{j\in B}\sum_{j\geq k} x_{ij}\right)_i$ *is a convergent sequence for each $B \in \mathcal{F}$.*
3. *The sequence $(\overline{x}^i)_i$ is convergent to some $\overline{x}^0 = (x_j)_j \in cs(X)$.*

*Proof.*
It is sufficient to prove that properties 2. and 3. are equivalent. In the same manner we can establish the equivalence between 1. and 3.

We first prove 2. ⇒ 3. Define $\overline{z}^i = \varphi \circ \phi(\overline{x}^i) = \left(\sum_j x_{ij}, -\sum_{j\geq 2} x_{ij}, -\sum_{j\geq 3} x_{ij}, \dots, -\sum_{j\geq n} x_{ij}, \dots\right) \in c_0(X)$ and write $\overline{z}^i = (z_{ik})_k$ with $z_{i1} = \sum_j x_{ij}$ and $z_{ik} = -\sum_{j\geq k} x_{ij}$ for $k > 1$. From 2. it follows that $\left(\sum_{k\in B} z_{ik}\right)_i$ is convergent for each $B \in \mathcal{F}$, where $\mathcal{F}$ has property $P_{c_0}$ , and so the sequence $(\overline{z}^i)_i$ and the family $\mathcal{F}$ verify condition 2. in Corollary 2.4. We can observe that if $1 \in B$ we obtain the same conclusion as $\left(\sum_j x_{ij}\right)_i$ is convergent. Then we can denote by $\overline{z}^0 \in c_0(X)$ the limit of $(\overline{z}^i)_i$. Define $\overline{x}^0 = \phi^{-1} \circ \varphi^{-1}(\overline{z}^0) \in cs(X)$, it is obvious that $\lim_i \overline{x}^i = \overline{x}^0$.

Conversely, define $\overline{z}^0 = \varphi \circ \phi(\overline{x}^0)$ and $\overline{z}^i = \varphi \circ \phi(\overline{x}^i)$ for each $i \in \mathbb{N}$. Corollary 2.4 allows us to consider a natural family $\mathcal{F}$ with property $P_{c_0}$ such that $\left(\sum_{j\in B} z_{ij}\right)_i$ is convergent for $B \in \mathcal{F}$. From the definition of $\phi$ and $\varphi$ it is easy to complete this argument. ∎

For Banach spaces with a Schauder basis, the following corollary asserts that every series which is $\mathcal{F}$-convergent in the topology $\sigma(X, M)$, where $M$ denotes the set of associated coordinate functionals, is actually convergent in $X$.

Corollary 3.10. *Let $X$ be a Banach space with a Schauder basis $\{a_i : i \in \mathbb{N}\}$ and coordinate functionals $M = \{g_i : i \in \mathbb{N}\}$ and let $\mathcal{F}$ be a natural family with property $P_1$. If $\sum_j x_j$ is a series in $X$ such that $\sum_{j\in B} x_j$ is $\sigma(X, M)$ convergent for $B \in \mathcal{F}$, then $\sum_j x_j$ is convergent in $X$.*

*Proof.* It is sufficient to observe that the matrix $(g_i(x_j)a_i)_{i,j}$ verifies the condition 3. in Theorem 3.6. ∎

Swartz [11] established the following result, which is based on the Antosik interchange theorem ([2]): Let $X$ be a Hausdorff topological vector space with a Schauder basis $\{a_i : i \in \mathbb{N}\}$ and coordinate functionals $M = \{g_i : i \in \mathbb{N}\}$. If $\sum_j x_j$ is $\sigma(X, M)$ subseries convergent, then $\sum_j x_j$ is subseries convergent in the original topology of $X$.

For Banach spaces, the following property ([1]) allows us to improve the above result:

Definition 3.11. *We say that a natural family $\mathcal{F}$ has property $S_1$ if for every pair $[(A_i)_i, (B_i)_i]$ of disjoint sequences of mutually disjoint sets in $\phi_0(\mathbb{N})$ there exists an infinite set $M \subseteq \mathbb{N}$ and $B \in \mathcal{F}$ satisfying $A_i \subseteq B$ and $B_i \subseteq B^c$ for each $i \in M$.*

Theorem 3.12. *Let $X$ be a Banach space with a Schauder basis $\{a_i : i \in \mathbb{N}\}$ and coordinate functionals $M = \{g_i : i \in \mathbb{N}\}$ and let $\mathcal{F}$ be a natural family with property $S_1$. If $\sum_j x_j$ is a series in $X$ such that $\sum_{j\in B} x_j$ is $\sigma(X, M)$ convergent for each $B \in \mathcal{F}$, then $\sum_j x_j$ is unconditionally convergent (uco).*

*Proof.* For $k, j \in \mathbb{N}$, let $z_{kj} = \sum_{i=1}^{k} g_i(x_j)a_i$. It is easy to check that the matrix $(z_{kj})_{k,j}$ is such that:

(i) $\left(\sum_{j\in B} z_{kj}\right)_k$ is convergent for each $B \in \mathcal{F}$.

(ii) $\sum_j z_{kj}$ is uco ([1]) for each $k \in \mathbb{N}$.

Then $\sum_j z_{kj}$ are uco uniformly on $k \in \mathbb{N}$ ([1]). The rest of the proof is obvious. ■

*Remark* 3.13 We now give some examples of natural families with property $P_1$ that are not $IQ$ $\sigma$-families. Define $\mathcal{F}_2 = \mathcal{B}_2 \cup \phi_0(\mathbb{N})$, where $\mathcal{B}_2$ is the family of the sets $A \subseteq \mathbb{N}$ such that both $A$ and $A^c$ contain infinitely many even numbers and odd numbers. Analysis similar to that in Remark 2.6 shows that $\mathcal{F}_2$ has property $P_1$. Let $f : \mathbb{N} \to \mathbb{N}$ be the map given by $f(i) = 1$ for $i \in \mathbb{N}$. If $(F_r)_r$ and $(m_r)_r$ are two sequences in $I_0(\mathbb{N})$ and $\mathbb{N}$, respectively, with $m_r \leq \inf F_r \leq \sup F_r < m_{r+1}$ for $r \in \mathbb{N}$, we now prove that there exist $B \in \mathcal{F}_2$ and an infinite set $M \subseteq \mathbb{N}$ which verify properties $(a)$ and $(b)$ in Definition 3.3.
Define $A = \bigcup_r F_r$, $B = \{2n : n \in \mathbb{N}\}$ and $C = \{2n-1 : n \in \mathbb{N}\}$. We consider two cases and proceed by induction in each of them:

(A) $A \cap B$ is an infinite set. In the first step we consider:

(i.1) $r_{11}, r_{12} \in \mathbb{N}$, with $r_{11} < r_{12}$, such that $F_{r_{11}}$ and $F_{r_{12}}$ have at least one even number.

(ii.1) $r_{13} \in \mathbb{N}$, $r_{13} \geq r_{12}$, satisfying $[m_{r_{13}}, m_{r_{13}+1}) \cap C \neq \emptyset$. Let $l_1$ be an element which belongs to the previous intersection set.

(iii.1) $r_{14} \in \mathbb{N}$, $r_{14} \geq r_{13}$, which verifies that there exists an odd number $n_1 > l_1$ which belongs to $[m_{r_{14}}, m_{r_{14}+1})$.

Define $A^1 = F_{r_{11}} \cup \{l_1\}$ and $M^1 = \{r_{11}\}$. Let us observe that $m_{r_1} \leq \inf F_{r_{11}} \leq \sup F_{r_{11}} < m_{r_{11}+1} \leq m_{r_{12}} \leq m_{r_{13}} \leq l_1$, and so $l_1 \notin [m_{r_{11}}, m_{r_{11}+1})$.

We now apply this argument again with the difference that $r_{21}$ must verify the inequality $r_{21} > r_{14}$. Hence, the second step allows us to obtain $r_{21}$, $r_{22}$, $r_{23}$, $r_{24} \in \mathbb{N}$, with $r_{21} < r_{22} \leq r_{23} \leq r_{24}$, such that:

(i.2) $F_{r_{21}}$ and $F_{r_{22}}$ have at least an even number.

(ii.2) $l_2 \in [m_{r_{23}}, m_{r_{23}+1}) \cap C$.

(iii.2) $n_2 > l_2$ and $n_2 \in [m_{r_{14}}, m_{r_{14}+1}) \cap C$.

Define $A^2 = F_{r_{21}} \cup \{l_2\}$ and $M^2 = \{r_{21}\}$. It is clear that $l_1 < n_1 < m_{r_{14}+1} \leq m_{r_{21}}$, and so $l_1$, $n_1 \notin [m_{r_{21}}, m_{r_{21}+1})$. Also, it can be checked that $l_2 \notin [m_{r_{21}}, m_{r_{21}+1})$.

The inductive argument we have sketched above gives us four strictly increasing sequences $(r_{i1})_i$, $(r_{i2})_i$, $(r_{i3})_i$ and $(r_{i4})_i$ in $\mathbb{N}$, with $r_{i1} < r_{i2} \leq r_{i3} \leq r_{i4} < r_{(i+1)1}$ for $i \in \mathbb{N}$, which verify the properties:

(a) $F_{r_{i1}}$ and $F_{r_{i2}}$ have at least one even number.

(b) For each $i \in \mathbb{N}$, there exist two odd numbers $l_i$ and $n_i$ with $n_i > l_i$, $l_i \in [m_{r_{i3}}, m_{r_{i3}+1})$ and $n_i \in [m_{r_{i4}}, m_{r_{i4}+1})$.

It can be checked that $l_k < n_k < m_{r_{i1}}$, for each $i \in \mathbb{N}$, $i > 1$, and $k \in \{1, 2, \dots, i-1\}$, and $l_i \geq m_{r_{i1}+1}$ for $i \in \mathbb{N}$.

Define $A^n = F_{r_{n1}} \cup \{l_n\}$ and $M^n = \{r_{n1}\}$ for $n \in \mathbb{N}$ and let us consider $B = \bigcup_n A^n$ and $M = \bigcup_n M^n$. It is easily seen that $B$ and $M$ verify properties $(a)$ and $(b)$ in Definition 3.3 and also $B \in \mathcal{F}_2$.

(B) $A \cap B$ is a finite set and $A \cap C$ is infinite. The proof is very similar to that in the previous case with the following differences, for each $i \in \mathbb{N}$: $(i)$ $r_{i1}$ and $r_{i2}$ must verify that $F_{r_{i1}}$ and $F_{r_{i2}}$ have at least one odd number instead of one even number; $(ii)$ $l_i$ and $n_i$ must be even numbers instead of odd numbers.

Let us observe that the union of the members of each subsequence of $([2n, 2n])_n$ does not belong to $\mathcal{F}_2$, and so this family is not an $IQ$ $\sigma$-family.

Moreover, if $Q_1$ is an infinite subset of $\mathbb{N}$ whose complement is also infinite, an analysis similar to that in Remark 2.6 allows us to adapt the above inductive argument in order to show that the following families have property $P_1$:

- $\mathcal{F}_\alpha = \{B \subseteq \mathbb{N} : B \cap Q_1 \text{ is infinite}\} \cup \phi_0(\mathbb{N})$
- $\mathcal{F}_\beta = \{B \subseteq \mathbb{N} : B \cap Q_1 \text{ and } B^c \cap Q_1 \text{ are infinite}\} \cup \phi_0(\mathbb{N})$.

Let $\{j_r\}$ be a strictly increasing sequence such that $Q_1^c = \{j_r : r \in \mathbb{N}\}$ and define $F_r = \{j_r\}$, for each $r \in \mathbb{N}$. Considering the sequence $(F_r)_r$, we conclude that $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ are not $IQ$ $\sigma$-families.

Let $Q_2$ and $Q_3$ be two infinite subsets of $\mathbb{N}$ with the property that at least one of them has an infinite complement. Similarly, the following families have property $P_1$ but are not $IQ$ $\sigma$-families.

- $\mathcal{F}_\gamma = \{B \subseteq \mathbb{N} : B \cap Q_2 \text{ and } B \cap Q_3 \text{ are infinite}\} \cup \phi_0(\mathbb{N})$.
- $\mathcal{F}_\theta = \{B \subseteq \mathbb{N} : B \cap Q_2, B \cap Q_3, B^c \cap Q_2 \text{ and } B^c \cap Q_3 \text{ are infinite }\} \cup \phi_0(\mathbb{N})$.

■

## 4 Convergence in $c(X)$

Let $X$ be a Banach space and let $c(X)$ be the space of convergent sequences in $X$, endowed with the norm

$$\|(x_n)_n\| = \sup_n \|x_n\|.$$

This space is linearly isometric to $cs(X)$. Let $\phi$ be the isometry from $cs(X)$ to $c(X)$ (see Remark 3.9). If $(\overline{x}^i)_i$ denotes a sequence in $c(X)$ (or $cs(X)$) we will write $\overline{x}^i = (x_{ij})_j$. From Corollary 3.8 we can now characterize the convergence of a sequence in $c(X)$, using the above notation.

Theorem 4.1. *Let $X$ be a Banach space and let $(\overline{x}^i)_i$ be a sequence in $c(X)$. The following statements are equivalent:*

1. *Assume that $(m_j)_j$ and $(n_j)_j$ are two sequences in $\mathbb{N}$ with $n_j < m_j \leq n_{j+1}$, for each $j \in \mathbb{N}$. Then there exists an infinite set $M \subseteq \mathbb{N}$ such that*
$$\left(\sum_{j\in M}(x_{im_j} - x_{in_j})\right)_i$$
*is a convergent sequence.*

2. *There exists a natural family with property $P_1$ such that $\left(\sum_{j\in B}(x_{ij} - x_{i(j-1)})\right)_i$ is convergent for each $B \in \mathcal{F}$ (let us consider $x_{i0} = 0$).*

3. *The sequence $(\overline{x}^i)_i$ converges to some $\overline{x}^0 = (x_j)_j \in c(X)$.*

*Proof.* It is sufficient to establish the equivalence between properties 1. and 3. Similarly, it can be shown that 2. and 3. are equivalent.

We first prove that 1. $\Rightarrow$ 3. For every $i \in \mathbb{N}$, write $\overline{x}^i = (x_{ij})_j$ and $\overline{y}^i = (y_{ij})_j = \phi^{-1}(\overline{x}^i)$. It is sufficient to show that the family $\mathcal{F} = \left\{B \subseteq \mathbb{N} : \left(\sum_{j\in B} y_{ij}\right)_i \text{ is convergent}\right\}$ is an $IQ$ $\sigma$-family (see Corollary 3.8). Suppose that $(F_j)_j$ is a sequence of intervals with $\sup F_j < \inf F_{j+1}$ and let $p_j = \inf F_j$ and $q_j = \sup F_j$, for $j \in \mathbb{N}$. Applying our hypothesis, the sequences $(p_j - 1)_j$ and $(q_j)_j$ allow us to obtain an infinite set $M \subseteq \mathbb{N}$ such that $\left(\sum_{l\in M}(x_{iq_l} - x_{i(p_l-1)})\right)_i = \left(\sum_{j\in B} y_{ij}\right)_i$, where $B = \bigcup_{l\in M} F_l$, is convergent. We can certainly assume that $\inf F_1 > 1$, for, if not, we replace $(F_j)_j$ by $(F_j)_{j>1}$. Let us observe that $\sum_{j\in F_l} y_{ij} = x_{iq_l} - x_{i(p_l-1))}$ if $p_l \neq 1$ and $\sum_{j\in F_l} y_{ij} = x_{iq_l}$ for $F_l = [1, q_l]$. This proves that $(\overline{y}^i)_i$ converges to some $\overline{y}^0 \in cs(X)$ (Corollary 3.8), and so that $\lim_i \overline{x}^i = \overline{x}^0$, where $\overline{x}^0 = \phi(\overline{y}^0)$.

Conversely, for every $i \in \mathbb{N}$ take $\overline{y}^i = (y_{ij})_j = \phi^{-1}(\overline{x}^i) = \phi^{-1}((x_{ij})_j)$ and $\overline{y}^0 = (y_j)_j = \phi^{-1}(\overline{x}^0)$. If $(m_j)_j$ and $(n_j)_j$ are two sequences satisfying $n_j < m_j \leq n_{j+1}$ and $F_j$ denotes the interval in $\mathbb{N}$ $[n_j + 1, m_j]$, for each $j \in \mathbb{N}$, from Corollary 3.8 it follows that there exists an infinite set $M \subseteq \mathbb{N}$ such that $\left(\sum_{j\in B} y_{ij}\right)_i$ is convergent, where $B = \bigcup_{l\in M} F_l$. Obviously, $\left(\sum_{l\in M}(x_{im_l} - x_{in_l})\right)_i$ also converges. ∎

*Remark* 4.2 From Corollary 2.4 and the isomorphism $\varphi$ between $c(x)$ and $c_0(X)$ (see Remark 3.9) we can obtain another characterization of the convergence of a sequence in $c(X)$. Let $(\overline{x}^i)_i$ be a sequence in $c(X)$, we will denote $\overline{x}^i = (x_{ij})_j$ and

$x_{i\infty} = \lim_j x_{ij}$ for $i \in \mathbb{N}$ . Analysis similar to that in the previous argument allows us to consider and prove the following result:

*Let $X$ be a Banach space and let $(\overline{x}^i)_i$ be a sequence in $c(X)$. The following statements are equivalent:*

1. *There exist $x_0 = \lim_i x_{i\infty}$ and a natural family $\mathcal{F}$ with property SC such that $\left(\sum_{j\in B}(x_{i(j-1)} - x_{i\infty})\right)_i$ is a convergent sequence for each $B \in \mathcal{F}$ (let us consider $x_{i0} = 0$).*

2. *There exist $x_0 = \lim_i x_{i\infty}$ and a natural family $\mathcal{F}$ with property $P_{c_0}$ such that $\left(\sum_{j\in B}(x_{i(j-1)} - x_{i\infty})\right)_i$ is a convergent sequence for each $B \in \mathcal{F}$ (let us consider $x_{i0} = 0$).*

3. *The sequence $(\overline{x}^i)_i$ is convergent to some $\overline{x}^0 = (x_j)_j \in c(X)$.*

*Proof.*
It is sufficient to prove that properties 2. and 3. are equivalent. In the same manner we can establish the equivalence between 1. and 3.

We first prove that 2. $\Rightarrow$ 3. Let us consider $\overline{y}^i = \varphi(\overline{x}^i) = (x_{i\infty}, x_{i1} - x_{i\infty}, \dots, x_{i(n-1)} - x_{i\infty} \dots)$. From 2. it is easy to check that $(\overline{y}^i)_i$ and the family $\mathcal{F}$ verify condition 2. in Corollary 2.4, and so there exists $\overline{y}^0 \in c_0(X)$ such that the sequence $(\overline{y}^i)_i$ converges to this element. We can observe that if $1 \in B$ we obtain the same conclusion as $(x_{i\infty})_i$ is a convergent sequence. It is enough to consider $\overline{x}^0 = \varphi^{-1}(\overline{y}^0)$, which satisfies $\overline{x}^0 = \lim_i \overline{x}^i$ in $c(X)$, in order to complete this argument.

Conversely, if property 3. is verified we will denote by $x_0$ the limit of the sequence $(x_j)_j = \overline{x}^0$. It need only be shown that $\lim_i x_{i\infty} = x_0$, as from Corollary 2.4 it is easy to complete the proof. For each $\varepsilon > 0$ let us consider $i_0 \in \mathbb{N}$ satisfying $\|\overline{x}^i - \overline{x}^0\| < \frac{\varepsilon}{3}$ for $i \geq i_0$. Fix $i \in \mathbb{N}$, $i \geq i_0$, and let $m_i \in \mathbb{N}$ be such that $\|x_{im_i} - x_{i\infty}\| < \frac{\varepsilon}{3}$ and $\|x_{m_i} - x_0\| < \frac{\varepsilon}{3}$. We are now in a position to check the following inequalities:

$$\|x_{i\infty} - x_0\| \leq \|x_{i\infty} - x_{im_i}\| + \|x_{im_i} - x_{m_i}\| + \|x_{m_i} - x_0\| < \varepsilon \text{ for } i \geq i_0. \qquad \blacksquare$$

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