Near polygons having a big sub near polygon isomorphic to $\mathbb{G}_n$

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Abstract

In [7] a new infinite class $\mathbb{G}_n$, $n \geq 0$, of near polygons was defined. The near $2n$-gon $\mathbb{G}_n$ has the property that it contains $\mathbb{G}_{n-1}$ as a big geodetically closed sub near polygon. In this paper, we determine all near $2n$-gons, $n \geq 4$, having $\mathbb{G}_{n-1}$ as a big geodetically closed sub near $2(n-1)$-gon under the additional assumption that every two points at distance 2 have at least two common neighbours. We will prove that such a near $2n$-gon is isomorphic to either $\mathbb{G}_n$, $\mathbb{G}_{n-1} \otimes \mathbb{G}_2$, or $\mathbb{G}_{n-1} \times L$ for some line L.

1 Definitions and Overview

1.1 Basic definitions

A *near polygon* is a partial linear space $(\mathcal{P}, \mathcal{L}, \mathrm{I})$, $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$, with the property that for every point $p \in \mathcal{P}$ and for every line $L \in \mathcal{L}$ there exists a unique point on L nearest to p. Here distances $\mathrm{d}(\cdot, \cdot)$ are measured in the collinearity graph. If n is the maximal distance between two points, then the near polygon is called a near $2n$-gon. A near 0-gon consists of one point, a near 2-gon is a line, and the class of near quadrangles coincides with the class of generalized quadrangles (GQ's) which were introduced by Tits in [10]. Near polygons themselves were introduced by Shult and Yanushka in [9] because of their relationship with certain line systems in Euclidean

*Postdoctoral Fellow of the Fund for Scientific Research - Flanders (Belgium)
Received by the editors September 2002.
Communicated by H. Van Maldeghem.
1991 *Mathematics Subject Classification* : 05B20, 51E12, 51E20.
Key words and phrases : near polygon, generalized quadrangle, hermitean variety.

spaces. Generalized $2n$-gons ([11]) and dual polar spaces ([3]) form two important classes of near polygons.

A set X of points in a near polygon $\mathcal{S}$ is called a *subspace* if every line meeting X in at least two points is completely contained in X. A subspace X is called *geodetically closed* if every point on a shortest path between two points of X is as well contained in X. Having a subspace X, we can define a subgeometry $\mathcal{S}_X$ of $\mathcal{S}$ by considering only those points and lines of $\mathcal{S}$ which are completely contained in X. If X is geodetically closed, then $\mathcal{S}_X$ clearly is a sub near polygon of $\mathcal{S}$. A geodetically closed sub near polygon $\mathcal{S}_X \neq \mathcal{S}$ is called *big* if every point outside $\mathcal{S}_X$ is collinear with a unique point of $\mathcal{S}_X$. If a geodetically closed sub near polygon $\mathcal{S}_X$ is a nondegenerate generalized quadrangle, then X (and often also $\mathcal{S}_X$) will be called a *quad*. Sufficient conditions for the existence of quads were given in [9]. For every point x of a near polygon $\mathcal{S}$, $\mathcal{L}(\mathcal{S}, x)$ denotes the incidence structure whose points, respectively lines, are the lines, respectively quads, through x (natural incidence). $\mathcal{L}(\mathcal{S}, x)$ is a partial linear space and called *the local space at* x. If X is a set of points in a near polygon, then $\mathcal{C}(X)$ denotes the unique minimal geodetically closed sub near polygon through X. ($\mathcal{C}(X)$ is the intersection of all geodetically closed sub near polygons through X.) We call $\mathcal{C}(X)$ the *geodetic closure* of X. If $X_1, \ldots, X_k$ are sets of points, then $\mathcal{C}(X_1 \cup \cdots \cup X_k)$ is also denoted by $\mathcal{C}(X_1, \ldots, X_k)$. If one of the arguments of $\mathcal{C}$ is a singleton $\{x\}$, we will often omit the braces and write $\mathcal{C}(\cdots, x, \cdots)$ instead of $\mathcal{C}(\cdots, \{x\}, \cdots)$.

A near polygon is said to have *order* (s, t) if every line is incident with exactly $s+1$ points and if every point is incident with exactly $t+1$ lines. A near polygon is called *dense* if every line is incident with at least three points and if every two points at distance 2 have at least two common neighbours. Dense near polygons satisfy several nice properties. By Lemma 19 of [2], every point of a dense near polygon $\mathcal{S}$ is incident with the same number of lines; we denote this number by $t_{\mathcal{S}}+1$. If x and y are two points of a dense near polygon, then by Theorem 4 of [2], $\mathcal{C}(x, y)$ is the unique geodetically closed sub near $[2 \cdot d(x, y)]$-gon through x and y. Geodetically closed sub near hexagons of a dense near polygon are called *hexes*. All local spaces of a dense near polygon are linear spaces. For every point x of a dense near $2n$-gon, a rank $n-1$ geometry $\mathcal{G}(\mathcal{S}, x)$ can be defined over the type set $\{1, \ldots, n-1\}$ whose i-objects are the geodetically closed sub near $2i$-gons through x and whose incidence relation is the symmetrized containment. The geometry $\mathcal{G}(\mathcal{S}, x)$ is called the *local geometry* at x. For $n = 3$ the notions of local space and local geometry are equivalent.

1.2 Overview

In [7] a new infinite class of dense near polygons was defined. The unique near $2n$-gon, $n \geq 0$, of this class was denoted by $\mathbb{G}_n$. The near polygon $\mathbb{G}_n$, $n \geq 1$, has the nice property that it contains $\mathbb{G}_{n-1}$ as a big geodetically closed sub near $2(n-1)$-gon, see Lemma 12 of [7]. Also the near polygon $\mathbb{G}_{n-1} \otimes \mathbb{G}_2$ (see Section 2.7) and the direct products $\mathbb{G}_{n-1} \times L$ (see Section 2.1) have this property. The examination whether this property is sufficient to characterize these near polygons led to the main theorem of the present paper.

Main Theorem. *Every near* $2n$*-gon* $\mathcal{S}$, $n \geq 4$, *which satisfies*

(A) *every two points at distance* 2 *have at least two common neighbours,*

(B) $\mathcal{S}$ *has a big geodetically closed sub near polygon isomorphic to* $\mathbb{G}_{n-1}$,

is isomorphic to either $\mathbb{G}_n$, $\mathbb{G}_{n-1} \otimes \mathbb{G}_2$ *or* $\mathbb{G}_{n-1} \times L$ *for some line* L.

The proof of our Main Theorem (Section 4) relies on the classification of dense near hexagons with three points on each line ([1]). We recall this classification in Section 3. But first we will give some notions and results which we will need later.

2 Some notions and results regarding near polygons

2.1 Direct product

Let $\mathcal{S}_1 = (\mathcal{P}_1, \mathcal{L}_1, \mathrm{I}_1)$ and $\mathcal{S}_2 = (\mathcal{P}_2, \mathcal{L}_2, \mathrm{I}_2)$ be two near polygons. A new near polygon $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ can be derived from $\mathcal{S}_1$ and $\mathcal{S}_2$. It is called the *direct product* of $\mathcal{S}_1$ and $\mathcal{S}_2$ and is denoted by $\mathcal{S}_1 \times \mathcal{S}_2$. We have: $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2$, $\mathcal{L} = (\mathcal{P}_1 \times \mathcal{L}_2) \cup (\mathcal{L}_1 \times \mathcal{P}_2)$, the point (x, y) of $\mathcal{S}_1 \times \mathcal{S}_2$ is incident with the line $(z, L) \in \mathcal{P}_1 \times \mathcal{L}_2$ if and only if $x = z$ and $y \ \mathrm{I}_2 \ L$, the point (x, y) of $\mathcal{S}_1 \times \mathcal{S}_2$ is incident with the line $(M, u) \in \mathcal{L}_1 \times \mathcal{P}_2$ if and only if $x \ \mathrm{I}_1 \ M$ and $y = u$. If $\mathcal{S}_i$, $i \in \{1, 2\}$, is a near $2n_i$-gon then the direct product $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$ is a near $2(n_1 + n_2)$-gon. Since $\mathcal{S}_1 \times \mathcal{S}_2 \cong \mathcal{S}_2 \times \mathcal{S}_1$ and $(\mathcal{S}_1 \times \mathcal{S}_2) \times \mathcal{S}_3 \cong \mathcal{S}_1 \times (\mathcal{S}_2 \times \mathcal{S}_3)$, also the direct product of $k \geq 3$ near polygons $\mathcal{S}_1, \ldots, \mathcal{S}_k$ is well-defined.

Theorem 1 (Theorem 1 of [2]) *Let* $\mathcal{S}$ *be a near polygon with the property that every two points at distance* 2 *have at least two common neighbours. If* $k \geq 2$ *different line sizes occur in* $\mathcal{S}$, *then* $\mathcal{S}$ *is isomorphic to a direct product* $\mathcal{S}_1 \times \cdots \times \mathcal{S}_k$ *of near polygons each of which has constant line size.*

2.2 Big geodetically closed sub near polygons

Let $\mathcal{S}$ be a near $2n$-gon. Recall that a geodetically closed sub near $2(n-1)$-gon $\mathcal{F}$ of $\mathcal{S}$ is called *big* if every point x outside $\mathcal{F}$ is collinear with a unique point $\pi(x)$ of $\mathcal{F}$. If $x \in \mathcal{F}$, then we put $\pi(x)$ equal to x. The map π is called the *projection on* $\mathcal{F}$. Properties of big geodetically closed sub near polygons are given in the following lemmas.

Lemma 1 *Let* $\mathcal{F}$ *be a big geodetically closed sub near polygon of* $\mathcal{S}$. *If* x *is a point outside* $\mathcal{F}$, *then* $d(x, y) = 1 + d(\pi(x), y)$ *for every point* $y \in \mathcal{F}$.

Proof. Since $\mathrm{d}(x, \pi(x)) = 1$, $\mathrm{d}(\pi(x), y) - 1 \leq \mathrm{d}(x, y) \leq \mathrm{d}(\pi(x), y) + 1$. If $\mathrm{d}(x, y) = \mathrm{d}(\pi(x), y) - 1$ or $\mathrm{d}(x, y) = \mathrm{d}(\pi(x), y)$, then the unique point z on the line $x\,\pi(x)$ nearest to y satisfies $\mathrm{d}(y, z) = \mathrm{d}(y, \pi(x)) - 1$. Hence $z \in \mathcal{C}(\pi(x), y) \subseteq \mathcal{F}$. Since $z, \pi(x) \in \mathcal{F}$, also the point x of the line $z\,\pi(x)$ belongs to $\mathcal{F}$, a contradiction. Hence $\mathrm{d}(x, y) = 1 + \mathrm{d}(\pi(x), y)$. ∎

Lemma 2 *Let $\mathcal{F}$ be a big geodetically closed sub near polygon of $\mathcal{S}$. If x and y are two collinear points outside $\mathcal{F}$ such that xy is disjoint with $\mathcal{F}$, then $d(\pi(x),\pi(y)) = 1$. For every line L outside $\mathcal{F}$, $\pi(L) := \{\pi(x)|x \mathrm{~I~} L\}$ is a line of $\mathcal{F}$.*

Proof. Since xy is disjoint with $\mathcal{F}$, $\mathrm{d}(x,\pi(y)) = 2$. Hence $\mathrm{d}(\pi(x),\pi(y)) = 1$ by Lemma 1. Since $\pi(L)$ is a set of mutually collinear points, there exists a line L' in $\mathcal{F}$ containing $\pi(L)$. Suppose that there exists a point $z \in L' \setminus \pi(L)$, then z has distance 2 to at least two points of L. Hence z is collinear with a unique point z' of L, contradicting $z \notin \pi(L)$. As a consequence $L' = \pi(L)$. ■

Lemma 3 *Let $\mathcal{F}$ be a big geodetically closed sub near polygon of $\mathcal{S}$. If x and y are two points outside $\mathcal{F}$ such that $\mathcal{C}(x,y)$ is disjoint with $\mathcal{F}$, then $d(x,y) = d(\pi(x),\pi(y))$.*

Proof. Every shortest path between x and y projects to a path of length $\mathrm{d}(x,y)$ between $\pi(x)$ and $\pi(y)$. Hence $\mathrm{d}(x,y)-2 \leq \mathrm{d}(\pi(x),\pi(y)) \leq \mathrm{d}(x,y)$. If $\mathrm{d}(x,y)-2 = \mathrm{d}(\pi(x),\pi(y))$ or $\mathrm{d}(x,y)-1 = \mathrm{d}(\pi(x),\pi(y))$, then $\mathrm{d}(x,\pi(y)) \leq \mathrm{d}(x,y)$. Hence there exists a unique point z on the line $y\,\pi(y)$ at distance $\mathrm{d}(x,y)-1$ from x. Now $z \in \mathcal{C}(x,y)$ since there exists a shortest path between x and y containing z. Since $z, y \in \mathcal{C}(x,y)$, also $\pi(y) \in \mathcal{C}(x,y)$, contradicting our assumption. Hence $\mathrm{d}(x,y) = \mathrm{d}(\pi(x),\pi(y))$. ■

By Lemmas 2 and 3, we then have:

Corollary 1 *Let $\mathcal{F}$ be a big geodetically closed sub near polygon of $\mathcal{S}$. Then every geodetically closed sub near polygon $\mathcal{F}'$ disjoint with $\mathcal{F}$ projects to a (not necessarily geodetically closed) sub near polygon $\pi(\mathcal{F}')$ of $\mathcal{F}$ isomorphic to $\mathcal{F}'$. Moreover, this projection preserves the distances.*

Lemma 4 (Lemma 4.5 of [1]) *If $\mathcal{F}$ is a big geodetically closed sub near $2(n-1)$-gon of a dense near $2n$-gon $\mathcal{S}$, $n \geq 2$, then the following are equivalent:*

(*a*) *$\mathcal{S} \cong \mathcal{F} \times L$;*

(*b*) *$t_{\mathcal{S}} = t_{\mathcal{F}} + 1$;*

(*c*) *every quad meeting $\mathcal{F}$ in a line is a grid.*

Lemma 5 *Let $\mathcal{S}$ be a dense near polygon, let $\mathcal{F}$ be a big geodetically closed sub near polygon of $\mathcal{S}$ and let x be an arbitrary point of $\mathcal{F}$. Then every geodetically closed sub near polygon $\mathcal{F}'$ through x either is contained in $\mathcal{F}$ or intersects $\mathcal{F}$ in a big geodetically closed sub near polygon of $\mathcal{F}$'*

Proof. Suppose that $\mathcal{F}' \not\subseteq \mathcal{F}$. Clearly $\mathcal{F} \cap \mathcal{F}'$ is geodetically closed. If y is a point of $\mathcal{F}' \setminus \mathcal{F}$, then y is collinear with a unique point $\pi(y)$ of $\mathcal{F}$. By Lemma 1, $\pi(y)$ lies on a shortest path between y and x. Hence $\pi(y) \in \mathcal{F} \cap \mathcal{F}'$. This proves that $\mathcal{F} \cap \mathcal{F}'$ is big in $\mathcal{F}'$. ■

Lemma 6 (Lemma 5 of [6]) *Let $\mathcal{S}$ be a dense near $2n$-gon, $n \geq 2$, let $\mathcal{F}$ denote a geodetically closed sub near $2(n-1)$-gon of $\mathcal{S}$ and let x denote an arbitrary point of $\mathcal{F}$. Then $\mathcal{F}$ is big in $\mathcal{S}$ if and only if every quad through x either is contained in $\mathcal{F}$ or intersects $\mathcal{F}$ in a line.*

Lemma 7 *For each $i \in \{1, 2\}$, let $\mathcal{S}_i$ be a dense near polygon, let $\mathcal{F}_i$ be a big geodetically closed sub near polygon of $\mathcal{S}_i$ and let x_i be a point of $\mathcal{F}_i$. Suppose that there exists an isomorphism ϕ from $\mathcal{F}_1$ to $\mathcal{F}_2$ mapping x_1 to x_2 and a bijection θ from the set of lines of $\mathcal{S}_1$ through x_1 to the set of lines of $\mathcal{S}_2$ through x_2 such that the following holds for all lines K, L and M through x_1:*

(*a*) *if K is contained in $\mathcal{F}_1$, then $\theta(K) = \phi(K)$;*

(*b*) *K, L and M are contained in a quad if and only if $\theta(K)$, $\theta(L)$ and $\theta(M)$ are contained in a quad.*

Then $\mathcal{G}(\mathcal{S}_1, x) \cong \mathcal{G}(\mathcal{S}_2, x_2)$.

Proof. Let $\mathcal{A}$ be a geodetically closed sub near polygon of $\mathcal{S}_1$ through x_1. If $\mathcal{A}$ is contained in $\mathcal{F}_1$, then we define $\mu(\mathcal{A}) := \phi(\mathcal{A})$. If $\mathcal{A}$ is not contained in $\mathcal{F}_1$, then we define $\mu(\mathcal{A}) = \mathcal{C}(\theta(K), \phi(\mathcal{A} \cap \mathcal{F}_1))$ where K is any line of $\mathcal{A}$ through x_1 not contained in $\mathcal{F}_1$. This is a good definition. If K' is another line with this property, then K, K' and $\mathcal{C}(K, K') \cap \mathcal{F}_1$ are contained in the same quad. By (a) and (b) also $\theta(K)$, $\theta(K')$ and $\phi(\mathcal{C}(K, K') \cap \mathcal{F}_1)$ are in the same quad and since $\phi(\mathcal{C}(K, K') \cap \mathcal{F}_1) \subseteq \phi(\mathcal{A} \cap \mathcal{F}_1)$, $\mathcal{C}(\theta(K), \phi(\mathcal{A} \cap \mathcal{F}_1)) = \mathcal{C}(\theta(K'), \phi(\mathcal{A} \cap \mathcal{F}_1))$. If $\mathcal{A}$ is a near $2i$-gon, $i \in \{1, \ldots, n-1\}$, then also $\mu(\mathcal{A})$ is a near $2i$-gon. Clearly, μ is an incidence preserving bijection between the set of objects of $\mathcal{G}(\mathcal{S}_1, x)$ and the set of objects of $\mathcal{G}(\mathcal{S}_2, x_2)$. ■

Suppose now that every line of $\mathcal{S}$ is incident with exactly three points. For every big geodetically closed sub near $2(n-1)$-gon $\mathcal{F}$ of $\mathcal{S}$, we can then define the following permutation $\mathcal{R}_\mathcal{F}$ on the point set of $\mathcal{S}$: if $x \in \mathcal{F}$, then we put $\mathcal{R}_\mathcal{F}(x) := x$; if $x \notin \mathcal{F}$, then we put $\mathcal{R}_\mathcal{F}(x)$ equal to unique third point of the line $x\,\pi(x)$. By Section 4 of [1], $\mathcal{R}_\mathcal{F}$ is an automorphism of order 2 of $\mathcal{S}$. We call $\mathcal{R}_\mathcal{F}$ the *reflection about* $\mathcal{F}$.

2.3 GQ's with three points on every line

If $\mathcal{S}$ is a generalized quadrangle with only lines of size 3, then one of the following possibilities occurs, see e.g. [8].

- $\mathcal{S}$ is degenerate: $\mathcal{S}$ consists of $k \geq 2$ lines of size 3 through a point.
- $\mathcal{S}$ is isomorphic to the (3×3)-grid, i.e. to the direct product of two lines of size 3. The (3×3)-grid has order $(2, 1)$.
- $\mathcal{S}$ is isomorphic to $W(2)$: the points and lines of $W(2)$ are the totally isotropic points and lines of a symplectic polarity in $\mathrm{PG}(3, 2)$. The generalized quadrangle $W(2)$ has order $(2, 2)$, or shortly order 2.

- $\mathcal{S}$ is isomorphic to $Q(5,2)$: the points and lines of $Q(5,2)$ are the points and lines lying on a nonsingular elliptic quadric in $\mathrm{PG}(5,2)$. The generalized quadrangle $Q(5,2)$ has order $(2,4)$.

In the sequel, a quad which is isomorphic to a grid, $W(2)$ or $Q(5,2)$ will be called a grid-quad, a $W(2)$-quad or a $Q(5,2)$-quad, respectively.

2.4 The point-quad relation

If $(x,\mathcal{Q})$ is a point-quad pair of a near polygon $\mathcal{S}$, then one of the following possibilities occurs, see Proposition 2.6 of [9].

(i) There exists a unique point x' in $\mathcal{Q}$ nearest to x and $\mathrm{d}(x,y)=\mathrm{d}(x,x')+\mathrm{d}(x',y)$ for every point $y\in\mathcal{Q}$. In this case the pair $(x,\mathcal{Q})$ is called *classical*.

(ii) The points in $\mathcal{Q}$ nearest to x form an ovoid of $\mathcal{Q}$, i.e. a set of points of $\mathcal{Q}$ intersecting each line in exactly on point. In this case the pair $(x,\mathcal{Q})$ is called *ovoidal*.

(iii) $\mathcal{Q}$ is thin and can be regarded as a complete bipartite graph. The set of points in $\mathcal{Q}$ nearest to x is a proper subset of size at least two of one of the two ovoids of $\mathcal{Q}$. In this case the pair $(x,\mathcal{Q})$ is called *thin-ovoidal*.

Lemma 8 *Let $\mathcal{S}$ be a dense near $2n$-gon with a $Q(5,2)$-quad $\mathcal{Q}$. If $\mathcal{F}$ is a geodetically closed sub near $2(n-1)$-gon of $\mathcal{S}$, then one of the following possibilities occurs:*

(a) $\mathcal{F}$ and $\mathcal{Q}$ are disjoint;

(b) $\mathcal{F}$ and $\mathcal{Q}$ intersect in a line;

(c) $\mathcal{Q}\subseteq\mathcal{F}$.

Proof. Suppose that $\mathcal{Q}$ and $\mathcal{F}$ have a point x in common. Since $\mathcal{F}$ is dense, it contains a point y at maximal distance $n-1$ from x, see e.g. [2]. Since $Q(5,2)$ has no ovoids, see e.g. Theorem 3.4.1 of [8], the pair $(y,\mathcal{Q})$ must be classical. If y' denotes the unique point of $\mathcal{Q}$ nearest to y, then $\mathrm{d}(y,z)=\mathrm{d}(y,y')+\mathrm{d}(y',z)$ for every point z of $\mathcal{Q}$ and hence $\mathrm{d}(y,y')\leq n-2$. Since $\mathrm{d}(y,x)=n-1$, $y'\neq x$. Since $\mathrm{d}(y,x)=\mathrm{d}(y,y')+\mathrm{d}(y',x)$, $y'\in\mathcal{C}(x,y)$ and hence $\mathcal{C}(x,y')\subseteq\mathcal{C}(x,y)=\mathcal{F}$. Since $x\neq y'$, $\mathcal{C}(x,y')$ is either $\mathcal{Q}$ or a line of $\mathcal{Q}$. This proves our lemma. ■

2.5 Admissible spreads in near polygons

For two lines K and L of a near polygon, let $\mathrm{d}(K,L)$ denote the minimal distance between a point of K and a point of L. By Lemma 1 of [2], one of the following possibilities occurs:

(a) there exist unique points $k\in K$ and $l\in L$ such that $\mathrm{d}(K,L)=\mathrm{d}(k,l)$;

(b) for every point $k \in K$ there exists a unique point $l \in L$ such that $\mathrm{d}(K, L) = \mathrm{d}(k, l)$.

If condition (b) is satisfied, then K and L are called *parallel.* A *spread* of a near polygon is a set of lines partitioning the point set. A spread is called *admissible* if every two lines of it are parallel. Clearly, every spread of a generalized quadrangle is admissible.

2.6 The near polygons $\mathbb{G}_n$

Let the vector space $V(2n, 4)$, $n \geq 1$, with base $B = \{\bar{e}_0, \ldots, \bar{e}_{2n-1}\}$ be equipped with the nonsingular Hermitian form $(\bar{x}, \bar{y}) = x_0 y_0^2 + x_1 y_1^2 + \ldots + x_{2n-1} y_{2n-1}^2$, let $H = H(2n-1, 4)$ denote the corresponding Hermitian variety in $\mathrm{PG}(2n-1, 4)$, and let ζ denote the Hermitian polarity associated with H. For every vector $\bar{x}$ of $V(2n, 4)$, we have $\bar{x} = \sum (\bar{x}, \bar{e}_i)\, \bar{e}_i$. The *support* S_p of a point $p = \langle \bar{x} \rangle$ of $\mathrm{PG}(2n-1, 4)$ is the set of all $i \in \{0, \ldots, 2n-1\}$ for which $(\bar{x}, \bar{e}_i) \neq 0$. The number $|S_p|$ is called the *weight* of p and is equal to the number of nonzero coordinates. A point of $\mathrm{PG}(2n-1, 4)$ belongs to H if and only if its weight is even. A subspace π on H is said to be *good* if it is generated by a (possibly empty) set $\mathcal{G}_\pi \subseteq H$ of points whose supports are two by two disjoint. If π is good, then $\mathcal{G}_\pi$ is uniquely determined. Let Y, respectively Y', denote the set of all good subspaces of dimension $n-1$, respectively $n-2$. With I denoting the reverse containment, we then can define an incidence structure $\mathbb{G}_n = (Y, Y', \mathrm{I})$. In [7] it was shown that $\mathbb{G}_n$ is a dense near $2n$-gon of order $(2, \frac{3n^2-n-2}{2})$ containing $\frac{3^n \cdot (2n)!}{2^n \cdot n!}$ points. The near polygon $\mathbb{G}_1$ is the line of size 3 and $\mathbb{G}_2$ is the generalized quadrangle $Q(5, 2)$. We recall some properties of $\mathbb{G}_n$, $n \geq 3$, see [7] for proofs.

- The near polygon $\mathbb{G}_n$, $n \geq 3$, has grid-quads, $W(2)$-quads and $Q(5, 2)$-quads.

- The automorphism group of $\mathbb{G}_n$, $n \geq 3$, acts transitively on the set of points. Hence, there exists a linear space $\mathcal{L}(\mathbb{G}_n)$ and a rank $n-1$ geometry $\mathcal{G}(\mathbb{G}_n)$ such that $\mathcal{L}(\mathbb{G}_n, x) \cong \mathcal{L}(\mathbb{G}_n)$ and $\mathcal{G}(\mathbb{G}_n, x) \cong \mathcal{G}(\mathbb{G}_n)$ for every point x of $\mathbb{G}_n$.

- The automorphism group $\mathrm{Aut}(\mathbb{G}_n)$, $n \geq 3$, has two orbits on the set of lines: the set of so-called *special lines* and the set of *ordinary lines.*

- Each point of $\mathbb{G}_n$ is contained in n special lines and $3\frac{n(n-1)}{2}$ ordinary lines. Each special line of $\mathbb{G}_n$ is contained in $n-1$ $Q(5, 2)$-quads, 0 $W(2)$-quads and $3\frac{(n-1)(n-2)}{2}$ grid-quads. Each ordinary line of $\mathbb{G}_n$ is contained in a unique $Q(5, 2)$-quad, $3(n-2)$ $W(2)$-quads and $3\frac{(n-2)(3n-7)}{2}$ grid-quads.

- If $L_1, \ldots, L_k$, are $k \geq 1$ special lines through a fixed point, then $\mathcal{C}(L_1, \ldots, L_k) \cong \mathbb{G}_k$. Conversely, if $\mathcal{F}$ is a geodetically closed sub near polygon of $\mathbb{G}_n$ isomorphic to $\mathbb{G}_k$, $k \geq 2$, and if x is an arbitrary point of $\mathcal{F}$, then precisely k from the n special lines through x are contained in $\mathcal{F}$.

- $\mathbb{G}_n$ has big geodetically closed sub near polygons isomorphic to $\mathbb{G}_{n-1}$ and every big geodetically closed sub near polygon of $\mathbb{G}_n$ is isomorphic to $\mathbb{G}_{n-1}$.

- For every $i \in \{0, \ldots, 2n-1\}$, the set B_i of those good subspaces of Y' which are contained in $\langle \bar{e}_i \rangle^{\zeta}$ is an admissible spread of $\mathbb{G}_n$. Conversely, every admissible spread of $\mathbb{G}_n$, $n \geq 3$, is of this form. The admissible spreads B_i, $i \in \{0, \ldots, 2n-1\}$, are precisely those spreads S of $\mathbb{G}_n$ which satisfy the following properties: (C1) every line of S is special, (C2) if a grid-quad $\mathcal{Q}$ of $\mathbb{G}_n$ contains one line of S, then it contains precisely 3 lines of S.

2.7 Glued near polygons

By "glueing" near polygons it is possible to derive new near polygons. This procedure was described in [4] for generalized quadrangles and in [5] for the general case. We recall the construction.

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two near polygons both with constant line size $s+1$, and suppose that their respective diameters d_1 and d_2 are at least 2. Let $S_i = \{L_1^{(i)}, \ldots, L_{\alpha_i}^{(i)}\}$, $i \in \{1, 2\}$, be an admissible spread of $\mathcal{A}_i$. In S_i, a special line $L_1^{(i)}$ is chosen which we will call the *base line*. For every $i \in \{1, 2\}$, for all $j, k \in \{1, \ldots, \alpha_i\}$ and for every $x \in L_j^{(i)}$, let $p_{j,k}^{(i)}(x)$ denote the unique point $L_k^{(i)}$ nearest to x. We put $\Phi_{j,k}^{(i)} := p_{k,1}^{(i)} \circ p_{j,k}^{(i)} \circ p_{1,j}^{(i)}$. For every $i \in \{1, 2\}$, the group $\Pi_{S_i}(L_1^{(i)}) := \langle \Phi_{j,k}^{(i)} | 1 \leq j, k \leq \alpha_i \rangle$ is called the *group of projectivities of* $L_1^{(i)}$ *with respect to* S_i.

For every bijection θ between $L_1^{(1)}$ and $L_1^{(2)}$, we consider the following graph Γ with vertex set $L_1^{(1)} \times S_1 \times S_2$. Two vertices $(x, L_{i_1}^{(1)}, L_{j_1}^{(2)})$ and $(y, L_{i_2}^{(1)}, L_{j_2}^{(2)})$ are adjacent if and only if exactly one of the following three conditions is satisfied:

(A) $L_{i_1}^{(1)} = L_{i_2}^{(1)}$, $L_{j_1}^{(2)} = L_{j_2}^{(2)}$ and $x \neq y$;

(B) $L_{j_1}^{(2)} = L_{j_2}^{(2)}$, $\mathrm{d}(L_{i_1}^{(1)}, L_{i_2}^{(1)}) = 1$ and $\Phi_{i_1,i_2}^{(1)}(x) = y$;

(C) $L_{i_1}^{(1)} = L_{i_2}^{(1)}$, $\mathrm{d}(L_{j_1}^{(2)}, L_{j_2}^{(2)}) = 1$ and $\Phi_{j_1,j_2}^{(2)} \circ \theta(x) = \theta(y)$.

By [5], the graph Γ has diameter $d_1 + d_2 - 1$ and every two adjacent vertices are contained in a unique maximal clique. Considering these maximal cliques as lines, we obtain a partial linear space $\mathcal{A}_1 \otimes \mathcal{A}_2$. If $\mathcal{A}_1 \otimes \mathcal{A}_2$ is a near polygon, then it is called a *glued near polygon*. This happens precisely when the condition in the following theorem is satisfied.

Theorem 2 (Theorem 14 of [5]) *The partial linear space $\mathcal{A}_1 \otimes \mathcal{A}_2$ is a glued near polygon if and only if the commutator $[\Pi_{S_1}(L_1^{(1)}), \theta^{-1}\Pi_{S_2}(L_1^{(2)})\theta]$ is the trivial group of permutations of $L_1^{(1)}$.*

Let us also mention the following result from [7].

Theorem 3 (Corollary 4 of [7]) *For all positive integers $m, n \geq 2$, there exists a unique glued near polygon of the form $\mathbb{G}_m \otimes \mathbb{G}_n$.*

3 Dense near hexagons with three points on each line

A near hexagon of order (s,t) is said to have parameters (s,t,T_2) if $T_2 = \{t_2(x,y) \mid \mathrm{d}(x,y)=2\}$. Here $t_2(x,y)+1$ denotes the number of common neighbours of x and y. If $s \geq 2$ and $0 \notin T_2$, then the near hexagon is dense. If there is a unique near hexagon with parameters (s,t,T_2), then we will denote it by $\mathbf{NH}(s,t,T_2)$.

Theorem 4 ([1]) *There are* 11 *dense near hexagons* $\mathcal{S}$ *with three points on each line. Each of these near hexagons is uniquely determined by its parameters:*

$\mathcal{S}$	big quads	other quads	local spaces
$\mathbf{NH}(2,2,\{1\})$	grid	—	$C_{2,2}$
$\mathbf{NH}(2,3,\{1,2\})$	grid, $W(2)$	—	$C_{2,3}$
$\mathbf{NH}(2,5,\{1,4\})$	grid, $Q(5,2)$	—	$C_{2,5}$
$\mathbf{NH}(2,5,\{1,2\})$	$W(2)$	grid	$\mathrm{PG}(2,2)^-$
$\mathbf{NH}(2,6,\{2\})$	$W(2)$	—	$\mathrm{PG}(2,2)$
$\mathbf{NH}(2,8,\{1,4\})$	$Q(5,2)$	grid	$C_{5,5}$
$\mathbf{NH}(2,11,\{1,2,4\})$	$Q(5,2)$	grid, $W(2)$	$\mathcal{L}(\mathbb{G}_3)$
$\mathbf{NH}(2,11,\{1\})$	—	grid	K_{12}
$\mathbf{NH}(2,14,\{2\})$	—	$W(2)$	$\mathrm{PG}(3,2)$
$\mathbf{NH}(2,14,\{2,4\})$	$Q(5,2)$	$W(2)$	$W(2)^+$
$\mathbf{NH}(2,20,\{4\})$	$Q(5,2)$	—	$\mathrm{PG}(2,4)$

We now define some of the above-mentioned linear spaces: (i) the (h,k)-cross $C_{h,k}$ is the unique linear space on $h+k-1$ vertices containing a line of length h and a line of length k which intersect in a point; all other lines have size 2, (ii) $\mathrm{PG}(2,2)^-$ is the linear space obtained from $\mathrm{PG}(2,2)$ by deleting a point, (iii) K_{12} is the complete graph on 12 vertices, (iv) $W(2)^+$ is the linear space obtained from $W(2)$ by regarding the 6 ovoids of $W(2)$ also as lines. (Notice that any two noncollinear points of $W(2)$ are contained in a unique ovoid.) The linear space $\mathcal{L}(\mathbb{G}_3)$ is the unique linear space on 12 points containing three lines of size 5, twelve lines of size 3 and nine lines of size 2. Removing the three points of $\mathcal{L}(\mathbb{G}_3)$ which are incident with two lines of size 5, we obtain the affine plane of order 3.

We have met some of the above-mentioned near hexagons before. With L denoting the line of size 3, we have $\mathbf{NH}(2,2,\{1\}) \cong L\times L\times L$, $\mathbf{NH}(2,3,\{1,2\}) \cong W(2)\times L$, $\mathbf{NH}(2,5,\{1,4\}) \cong Q(5,2)\times L$, $\mathbf{NH}(2,8,\{1,4\}) \cong \mathbb{G}_2\otimes\mathbb{G}_2$ and $\mathbf{NH}(2,11,\{1,2,4\}) \cong \mathbb{G}_3$.

4 Proof of the Main Theorem

In this section we will determine all near $2n$-gons $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$, $n\geq 4$, that satisfy the following properties:

(A) every two points at distance 2 have at least two common neighbours;

(B) $\mathcal{S}$ has a big geodetically closed sub near $2(n-1)$-gon $\mathcal{F}$ isomorphic to $\mathbb{G}_{n-1}$.

We will prove by induction that every such $\mathcal{S}$ is isomorphic to either $\mathbb{G}_n$, $\mathbb{G}_{n-1} \otimes \mathbb{G}_2$ or $\mathbb{G}_{n-1} \times L$ for some line L. Every line of $\mathcal{F}$ is incident with three points. If not all lines of $\mathcal{S}$ are incident with three points, then by Theorem 1, $\mathcal{S} \cong \mathcal{A} \times \mathcal{B}$ where $\mathcal{A}$ is a near polygon with only lines of size 3 and where $\mathcal{B}$ is a near polygon with no lines of size 3. Since $\mathcal{A}$ contains a sub near polygon isomorphic to $\mathbb{G}_{n-1}$, we necessarily have $\mathcal{A} \cong \mathbb{G}_{n-1}$ and $\mathcal{B} \cong L$ for some line L with $|L| \neq 3$. Hence $\mathcal{S} \cong \mathbb{G}_{n-1} \times L$ and we are done. From now on we assume that every line of $\mathcal{S}$ is incident with exactly $s + 1 = 3$ points. The near $2n$-gon $\mathcal{S}$ is then dense and geodetically closed sub near polygons exist. We put $t + 1 = t_{\mathcal{S}} + 1$. If $t = t_{\mathcal{F}} + 1$, then $\mathcal{S} \cong \mathbb{G}_{n-1} \times L$, $|L| = 3$, by Lemma 4. We suppose therefore that $t > t_{\mathcal{F}} + 1$.

Lemma 9 *If a $Q(5,2)$-quad $\mathcal{Q}$ intersects $\mathcal{F}$ in a line, then this line is a special line of $\mathcal{F} \cong \mathbb{G}_{n-1}$.*

Proof. Suppose that $L := \mathcal{Q} \cap \mathcal{F}$ is an ordinary line of $\mathcal{F}$. By Section 2.6, L is contained in a $W(2)$-quad $\mathcal{R} \subset \mathcal{F}$. By Lemma 5, the $W(2)$-quad $\mathcal{R}$ is big in the hex $\mathcal{H} := \mathcal{C}(\mathcal{Q}, \mathcal{R})$. By Theorem 4, none of the near hexagons with a big $W(2)$-quad contains a $Q(5,2)$-quad. This contradicts the fact that $\mathcal{Q} \subset \mathcal{H}$. Hence L is a special line of $\mathcal{F}$. ■

Lemma 10 *No hex $\mathcal{H}$ isomorphic to* $\mathbf{NH}(2,11,\{1\})$, $\mathbf{NH}(2,14,\{2\})$, $\mathbf{NH}(2,14,\{2,4\})$ *or* $\mathbf{NH}(2,20,\{4\})$ *meets $\mathcal{F}$.*

Proof. Suppose the contrary. By Lemma 5, $\mathcal{H} \cap \mathcal{F}$ is a big quad of $\mathcal{H}$. By Theorem 4, we then have: (i) $\mathcal{H} \cong \mathbf{NH}(2,14,\{2,4\})$ or $\mathcal{H} \cong \mathbf{NH}(2,20,\{4\})$, and (ii) $\mathcal{Q} := \mathcal{H} \cap \mathcal{F} \cong Q(5,2)$. By Section 2.6, the $Q(5,2)$-quad $\mathcal{Q}$ contains an ordinary line K of $\mathcal{F}$. By (i), $\mathcal{H}$ has a $Q(5,2)$-quad through K different from $\mathcal{Q}$. This quad contradicts Lemma 9. ■

Lemma 11 *Every point x of $\mathcal{F}$ is contained in a $Q(5,2)$-quad which intersects $\mathcal{F}$ in a line. Hence $t \geq t_{\mathcal{F}} + 4$.*

Proof. Since $t > t_{\mathcal{F}} + 1$, there exist two lines K and L through x not contained in $\mathcal{F}$. Since $\mathcal{F}$ is big in $\mathcal{S}$, $\mathcal{C}(K, L)$ intersects $\mathcal{F}$ in a line M; hence $\mathcal{C}(K, L) \cong W(2)$ or $\mathcal{C}(K, L) \cong Q(5,2)$. Suppose that $\mathcal{C}(K, L) \cong W(2)$. By Section 2.6, there exists a $Q(5,2)$-quad $\mathcal{Q} \subset \mathcal{F}$ through M. The hex $\mathcal{H} := \mathcal{C}(K, \mathcal{R})$ contains a $Q(5,2)$-quad and a $W(2)$-quad. By Theorem 4 and Lemma 10, $\mathcal{H}$ is isomorphic to $\mathbb{G}_3$ and hence contains a $Q(5,2)$-quad through x different from $\mathcal{Q}$. This proves our lemma. ■

First Case: $t = t_{\mathcal{F}} + 4$

Let P_2 denote the set of all $Q(5,2)$-quads meeting $\mathcal{F}$ in a line. By Lemma 11 and the fact that $t = t_{\mathcal{F}} + 4$, it follows that every point $x \in \mathcal{F}$ is contained in a unique element of P_2. If y is an arbitrary point outside $\mathcal{F}$, then $\mathcal{Q}_y := \mathcal{Q}_{\pi(y)}$ is the unique element of P_2 through y. Hence P_2 is a partition of the point set of $\mathcal{S}$ in $Q(5,2)$-quads. Clearly the set $S_1 := \{\mathcal{Q} \cap \mathcal{F} | \mathcal{Q} \in P_2\}$ is a spread S_1 of $\mathcal{F}$.

Lemma 12 *The spread* S_1 *is an admissible spread of* $\mathcal{F}$.

Proof. Since $\mathcal{F} \cong \mathbb{G}_{n-1}$, we need to verify the two conditions (C1) and (C2) mentioned in Section 2.6. Property (C1) is exactly Lemma 9. We now proof that also (C2) is satisfied. Let K be an arbitrary line of S_1, let $\mathcal{Q}$ denote the unique quad of P_2 through K and let $\mathcal{R}$ be an arbitrary grid-quad of $\mathcal{F}$ through K. The hex $\mathcal{H} := \mathcal{C}(\mathcal{Q}, \mathcal{R})$ has a $Q(5,2)$-quad and a big grid-quad and hence is isomorphic to $Q(5,2) \times L$ by Theorem 4. As a consequence $\mathcal{H}$ contains three quads of P_2 and the two lines of $\mathcal{R}$ disjoint from K also belong to S_1. ∎

Lemma 13 *Every geodetically closed sub near* $2(n-1)$*-gon isomorphic to* $\mathbb{G}_2 \otimes \mathbb{G}_{n-2}$ *meets* $\mathcal{F}$.

Proof. Let $\mathcal{F}'$ be a geodetically closed sub near $2(n-1)$-gon isomorphic to $\mathbb{G}_2 \otimes \mathbb{G}_{n-2}$ and disjoint from $\mathcal{F}$. The near hexagon $\mathcal{S}$ has $v_{\mathcal{S}} = (1+2\cdot(t-t_{\mathcal{F}}))\cdot|\mathcal{F}| = \frac{3^{n+1}\cdot(2n-2)!}{2^{n-1}\cdot(n-1)!}$ points. The total number of points at distance at most 1 from $\mathcal{F}'$ equals $(1+2(t-t_{\mathcal{F}'}))\cdot|\mathcal{F}'|$. Since this number is precisely $v_{\mathcal{S}}$, also $\mathcal{F}'$ is big in $\mathcal{S}$. Applying Corollary 1 twice, we see that $\mathcal{F} \cong \mathcal{F}'$. From $\frac{3(n-1)^2-(n-1)-2}{2} = t_{\mathcal{F}} = t_{\mathcal{F}'} = \frac{3(n-2)^2-(n-2)-2}{2} + 4$, it then follows that $n = 3$, but this contradicts our assumption $n \geq 4$. ∎

Lemma 14 *Every point* y *of* $\mathcal{S}$ *is contained in a unique big geodetically closed sub near polygon* $\mathcal{F}_y$ *satisfying:*

(*i*) $\mathcal{F}_y \cong \mathcal{F}$;

(*ii*) $\mathcal{F}_y = \mathcal{F}$ *or* $\mathcal{F}_y \cap \mathcal{F} = \emptyset$.

Proof. Suppose that y is contained in two such sub near polygons $\mathcal{F}_1$ and $\mathcal{F}_2$. Since $\mathcal{F}_3 := \mathcal{F}_1 \cap \mathcal{F}_2$ is big in $\mathcal{F}_1$, $\mathcal{F}_3 \cong \mathbb{G}_{n-2}$ by Section 2.6. Hence $t \geq t_{\mathcal{F}_1} + t_{\mathcal{F}_2} - t_{\mathcal{F}_3}$ or $t_{\mathcal{F}_2} - t_{\mathcal{F}_3} \leq 4$. Since $t_{\mathcal{F}_2} - t_{\mathcal{F}_3} = 3n-5$, $n \leq 3$, a contradiction. So, it suffices to show that y is contained in at least one big geodetically closed sub near polygon satisfying (i) and (ii). This trivially holds if $y \in \mathcal{F}$, so we suppose that $y \notin \mathcal{F}$. By Lemma 9, $\mathcal{Q}_y$ intersects $\mathcal{F}$ in a special line K. If $L_1, \ldots, L_{n-2}$ denote the other special lines of $\mathcal{F}$ through $\pi(y)$, then $\mathcal{F}_4 := \mathcal{C}(L_1, \ldots, L_{n-2})$ is isomorphic to $\mathbb{G}_{n-2}$. Put $\mathcal{F}_5 := \mathcal{C}(L_1, \ldots, L_{n-2}, y\,\pi(y))$. Since $t_{\mathcal{F}_5} = t_{\mathcal{F}_4} + 1$, $\mathcal{F}_5 \cong \mathcal{F}_4 \times L$. Hence y is contained in a geodetically closed sub near $2(n-2)$-gon $\mathcal{F}'_y$ isomorphic to $\mathbb{G}_{n-2}$. By Lemma 8 every geodetically closed sub near $2(n-1)$-gon through $\mathcal{F}'_y$ intersect $\mathcal{Q}_y$ in a line. Hence there are exactly five geodetically closed sub near $2(n-1)$-gons through $\mathcal{F}'_y$. One of them is $\mathcal{F}_5$. Let $\mathcal{F}_6$ denote one of the four others. The projection of $\mathcal{F}_6$ on $\mathcal{F}$ is distance-preserving and since the projection $\mathcal{C}(L_1, \ldots, L_{n-2})$ of $\mathcal{F}'_y$ is big in $\mathcal{F}$, also $\mathcal{F}'_y$ is big in $\mathcal{F}_6$. If $n = 4$, then $\mathcal{F}'_y \cong Q(5,2)$ and hence $\mathcal{F}_6 \cong \mathbb{G}_3$ or $\mathcal{F}_6 \cong \mathbb{G}_2 \times L$ by Theorem 4, Lemma 10 and Lemma 13. If $n \geq 5$, then $\mathcal{F}'_y \cong \mathbb{G}_{n-2}$ and hence $\mathcal{F}_6 \cong \mathbb{G}_{n-1}$ or $\mathcal{F}_6 \cong \mathbb{G}_{n-2} \times L$ by the induction hypothesis and Lemma 13. Suppose now that all the five geodetically closed sub near $2(n-1)$-gons through $\mathcal{F}'_y$ are isomorphic to $\mathbb{G}_{n-2} \times L$. Then $t = t_{\mathcal{F}'_y} + 5$ or $t_{\mathcal{F}} = t_{\mathcal{F}'_y} + 1$, a contradiction since $t_{\mathcal{F}} - t_{\mathcal{F}'_y} = 3n-5$ and $n \geq 4$. Hence there exists a geodetically closed sub near $2(n-1)$-gon through $\mathcal{F}'_y$ isomorphic to $\mathbb{G}_{n-1}$. Our lemma now follows since $y \in \mathcal{F}'_y$. ∎

The geodetically closed sub near $2(n-1)$-gons $\mathcal{F}_y$, $y \in \mathcal{P}$, determine a partition P_1 of $\mathcal{S}$ in sub near polygons isomorphic to $\mathbb{G}_{n-1}$. Every quad of P_2 intersects each sub near polygon of P_1 in a line and the set S of all lines obtained this way is a spread of $\mathcal{S}$.

Lemma 15 *The spread S is admissible.*

Proof. Take two arbitrary lines L_1 and L_2 of S. Let $\mathcal{F}'$ denote the unique elements of P_1 through L_1 and let $\mathcal{Q}'$ denote the unique element of P_2 through L_2. If L_2 is contained in $\mathcal{F}'$, then L_1 and L_2 are parallel by Lemma 12 (applied to $\mathcal{F}'$ instead of $\mathcal{F}$). If L_2 is not contained in $\mathcal{F}'$, then by Lemma 1 $\mathrm{d}(x, L_1) = 1 + \mathrm{d}(\pi_{\mathcal{F}'}(x), L_1)$ for every point x on L_2. Since $\pi_{\mathcal{F}'}(L_2) = \mathcal{Q}' \cap \mathcal{F}'$ belongs to S, $\pi_{\mathcal{F}'}(L_2)$ and L_1 are parallel. Hence, $\mathrm{d}(x, L_1)$ is independent of the chosen point $x \in L_2$. This proves that L_1 and L_2 are parallel and that S is admissible. ∎

Theorem 5 *The near polygon $\mathcal{S}$ is isomorphic to $\mathbb{G}_2 \otimes \mathbb{G}_{n-1}$.*

Proof. Put $\mathcal{A}_1 := \mathcal{F}$ and let $\mathcal{A}_2$ be any quad of P_2. Above we defined the admissible spread S_1 of $\mathcal{A}_1$. If we intersect $\mathcal{A}_2$ with all elements of P_1, then we obtain an admissible spread S_2 in $\mathcal{A}_2$. We consider the line $K := \mathcal{A}_1 \cap \mathcal{A}_2$ as base line in both S_1 and S_2 and we put θ equal to the trivial permutation of K. With these choices, we can define a glued incidence structure $\mathcal{A}_1 \otimes \mathcal{A}_2$, see Section 2.7. We will prove that $\mathcal{S} \cong \mathcal{A}_1 \otimes \mathcal{A}_2$. For every point x of $\mathcal{S}$, we put $\phi(x) := (x', \mathcal{Q}_x \cap \mathcal{A}_1, \mathcal{F}_x \cap \mathcal{A}_2)$ where x' denotes the unique element of K nearest to x. Clearly $\phi(x)$ is a point of $\mathcal{A}_1 \otimes \mathcal{A}_2$. Conversely, suppose that (y, L_1, L_2) is a point of $\mathcal{A}_1 \otimes \mathcal{A}_2$. Let $\mathcal{Q}'$ denote the unique element of P_2 through L_1, let $\mathcal{F}'$ denote the unique element of P_1 through L_2 and let x denote the unique point on the line $\mathcal{Q}' \cap \mathcal{F}'$ nearest to y. Since K and $\mathcal{Q}' \cap \mathcal{F}'$ are parallel, $\phi(x) := (y, L_1, L_2)$. Obviously, x is the only point of $\mathcal{S}$ which is mapped to (y, L_1, L_2) by ϕ. Hence ϕ is a bijection between the point sets of $\mathcal{S}$ and $\mathcal{A}_1 \otimes \mathcal{A}_2$. Take now two collinear points x and y in $\mathcal{S}$ and put $\phi(x) = (x', L_1, L_2)$ and $\phi(y) = (y', M_1, M_2)$. If the line xy belongs to S, then $L_1 = M_1$, $L_2 = M_2$ and $x' \neq y'$; hence also $\phi(x)$ and $\phi(y)$ are collinear. If $xy \subset \mathcal{F}_x$ and $xy \not\subset \mathcal{Q}_x$, then $L_2 = M_2$ and $\mathrm{d}(L_1, M_1) = 1$ since $\mathrm{d}(\pi(x), \pi(y)) = 1$ by Lemma 2. By Lemma 1, x' (resp. y') is the unique point of K nearest to $\pi(x)$ (resp. $\pi(y)$). The condition $\mathrm{d}(\pi(x), \pi(y)) = 1$ is equivalent with condition (B) of Section 2.7. Hence $\phi(x)$ and $\phi(y)$ are collinear points in $\mathcal{A}_1 \otimes \mathcal{A}_2$. Finally, suppose that $xy \not\subset \mathcal{F}_x$ and $xy \subset \mathcal{Q}_x$. Clearly $L_1 = M_1$. Let x'' and y'' denote the unique points of $\mathcal{A}_2$ nearest to x and y. Notice that these points exist since $(x, \mathcal{A}_2)$ and $(y, \mathcal{A}_2)$ are classical. (Recall that $\mathcal{A}_2 \cong Q(5, 2)$ has no ovoids.) Now, $\mathcal{F}_x$ and $\mathcal{F}_y$ are big and different, and so the projection of $\mathcal{F}_x$ on $\mathcal{F}_y$ is an isomorphism. As a consequence, the unique point x'' of $\mathcal{A}_2 \cap \mathcal{F}_x$ nearest to x is mapped by this isomorphism on the unique point y'' of $\mathcal{A}_2 \cap \mathcal{F}_y$ nearest to y. Hence $\mathrm{d}(x'', y'') = 1$ and $\mathrm{d}(L_2, M_2) = 1$. The condition $\mathrm{d}(x'', y'') = 1$ is equivalent with condition (C) of Section 2.7. Hence $\phi(x)$ and $\phi(y)$ are collinear points in $\mathcal{A}_1 \otimes \mathcal{A}_2$. Summarizing we find that ϕ is an adjacency preserving map between the collinearity graphs of $\mathcal{S}$ and $\mathcal{A}_1 \otimes \mathcal{A}_2$. Since both graphs have the same valency, they are isomorphic. As a consequence also $\mathcal{S}$ and $\mathcal{A}_1 \otimes \mathcal{A}_2$ are isomorphic. (Notice that the lines of a near polygon correspond with the maximal cliques of its collinearity graph.) The theorem now follows from Theorem 3. ∎

Second Case: $t > t_{\mathcal{F}} + 4$

Put $\delta := t - t_{\mathcal{F}}$.

Lemma 16 *We have* $\delta \leq 3n - 2$. *If equality holds, then no hex isomorphic to* $\mathbb{G}_2 \otimes \mathbb{G}_2$ *meets* $\mathcal{F}$.

Proof. By Lemmas 9 and 11 there exists a $Q(5,2)$-quad $\mathcal{Q}$ which intersects $\mathcal{F}$ in a special line K. By Theorem 4 and Lemma 10, every hex $\mathcal{H}$ through $\mathcal{Q}$ is isomorphic to either $\mathbb{G}_2 \times L$, $\mathbb{G}_2 \otimes \mathbb{G}_2$ or $\mathbb{G}_3$. In the first case $\mathcal{H} \cap \mathcal{F}$ is a grid. In the two other cases $\mathcal{H} \cap \mathcal{F}$ is a $Q(5,2)$-quad. Let λ_1, respectively λ_2, denote the number of hexes through $\mathcal{Q}$ which are isomorphic to $\mathbb{G}_2 \otimes \mathbb{G}_2$, respectively $\mathbb{G}_3$. By Section 2.6, $\mathcal{F}$ has $n-2$ $Q(5,2)$-quads through K and hence $\lambda_1 + \lambda_2 = n-2$. Counting over all hexes $\mathcal{H}$ through $\mathcal{Q}$, we find that $\delta = t_{\mathcal{Q}} + \sum(t_{\mathcal{H}} - t_{\mathcal{Q}} - t_{\mathcal{H}\cap\mathcal{F}}) = 4 + 3\lambda_2 \leq 4 + 3(n-2) = 3n-2$. The lemma now immediately follows. ∎

Lemma 17 *If a* $W(2)$*-quad* $\mathcal{Q}$ *intersects* $\mathcal{F}$ *in a line, then this line is an ordinary line of* $\mathcal{F} \cong \mathbb{G}_{n-1}$.

Proof. Suppose that $\mathcal{Q} \cap \mathcal{F}$ is a special line and let $x \in \mathcal{Q} \cap \mathcal{F}$. If $\mathcal{R}$ is one of the $n-2$ $Q(5,2)$-quads of $\mathcal{F}$ through $\mathcal{Q} \cap \mathcal{F}$, then the hex $\mathcal{C}(\mathcal{Q},\mathcal{R})$ has $W(2)$-quads and $Q(5,2)$-quads. By Theorem 4 and Lemma 10, it then follows that $\mathcal{C}(\mathcal{Q},\mathcal{R}) \cong \mathbb{G}_3$. Hence the hex $\mathcal{C}(\mathcal{Q},\mathcal{R})$ contains exactly five lines through x which are not contained in $\mathcal{Q} \cup \mathcal{R}$. Summing over all possible $\mathcal{R}$, we find that $\delta \geq 2 + 5(n-2) = 5n-8$. Together with $\delta \leq 3n-2$, this implies that $n \leq 3$, a contradiction. Hence $\mathcal{Q} \cap \mathcal{F}$ is an ordinary line. ∎

Lemma 18 *Every point* x *of* $\mathcal{F}$ *is contained in a* $W(2)$*-quad which intersects* $\mathcal{F}$ *in a line.*

Proof. By Lemma 11, there exists a $Q(5,2)$-quad $\mathcal{Q}$ through x intersecting $\mathcal{F}$ in a line. Since $t > t_{\mathcal{F}} + 4$, there exists a line K through x not contained in $\mathcal{Q} \cup \mathcal{F}$. By Theorem 4 and Lemma 10, the hex $\mathcal{H} = \mathcal{C}(\mathcal{Q},K)$, which intersects $\mathcal{F}$ in a big quad, is isomorphic to $\mathbb{G}_3$. The required $W(2)$-quad can now be chosen in the hex $\mathcal{H}$. ∎

Lemma 19 *We have* $\delta \geq 3n-2$. *If equality holds, then no hex isomorphic to* $\mathbf{NH}(2,6,\{2\})$ *meets* $\mathcal{F}$.

Proof. Let $\mathcal{Q}$ denote a $W(2)$-quad intersecting $\mathcal{F}$ in an ordinary line K. By Section 2.6, K is contained in a unique $Q(5,2)$-quad and $3(n-3)$ $W(2)$-quads of $\mathcal{F}$. If $\mathcal{T}$ is the unique $Q(5,2)$-quad, then the hex $\mathcal{H} := \mathcal{C}(\mathcal{Q},\mathcal{T})$ is isomorphic to $\mathbb{G}_3$. If $\mathcal{T}$ is one of the $3(n-3)$ $W(2)$-quads of $\mathcal{F}$ through K, then $\mathcal{H} = \mathcal{C}(\mathcal{Q},\mathcal{T})$ is isomorphic to either $\mathbf{NH}(2,5,\{1,2\})$ or $\mathbf{NH}(2,6,\{2\})$. Hence $\delta = t_{\mathcal{Q}} + \sum(t_{\mathcal{H}} - t_{\mathcal{Q}} - t_{\mathcal{H}\cap\mathcal{F}}) \geq 2 + 5 + 3(n-3) = 3n-2$. The lemma now immediately follows. ∎

From Lemmas 16 and 19, we then have:

Corollary 2 *The following holds:*

- $\delta = 3n-2$, $t = \delta + t_{\mathcal{F}} = \frac{3n^2-n-2}{2}$, $|\mathcal{P}| = (2\delta+1)\cdot|\mathcal{F}| = \frac{3^n\cdot(2n)!}{2^n\cdot n!}$ *and* $|\mathcal{L}| = \frac{|\mathcal{P}|\cdot(t+1)}{3} = \frac{3^{n-1}(2n)!(3n-1)}{2^{n+1}(n-1)!}$;
- *no hex isomorphic to* $\mathbb{G}_2 \otimes \mathbb{G}_2$ *meets* $\mathcal{F}$;
- *no hex isomorphic to* $\mathbf{NH}(2,6,\{2\})$ *meets* $\mathcal{F}$.

Lemma 20 (*a*) *Every special line* L *of* $\mathcal{F} \cong \mathbb{G}_{n-1}$ *is contained in a unique* $Q(5,2)$*-quad which is not contained in* $\mathcal{F}$.

(*b*) *Let* $x \in \mathcal{F}$. *All the* $Q(5,2)$*-quads through* x *which are not contained in* $\mathcal{F}$ *have a common line* A_x *in common.*

Proof.

(a) Suppose that the line L is contained in two such $Q(5,2)$-quads $\mathcal{Q}$ and $\mathcal{R}$. The hex $\mathcal{C}(\mathcal{Q},\mathcal{R})$ intersects $\mathcal{F}$ in a big quad, which is necessarily isomorphic to $Q(5,2)$. The line L of $\mathcal{C}(\mathcal{Q},\mathcal{R})$ is then contained in at least three $Q(5,2)$-quads and hence $\mathcal{C}(\mathcal{Q},\mathcal{R})$ must be isomorphic to $\mathbf{NH}(2,20,\{4\})$, contradicting Lemma 10. Hence L is contained in at most one $Q(5,2)$-quad which is not contained in $\mathcal{F}$. We will now prove that L is contained in a unique such $Q(5,2)$-quad. Let $x \in L$ and let $\mathcal{T}$ denote an arbitrary $Q(5,2)$-quad through x which intersects $\mathcal{F}$ in a special line. We may suppose that $L \neq \mathcal{T} \cap \mathcal{F}$. The hex $\mathcal{C}(\mathcal{T}, L)$ has at least two $Q(5,2)$ quads through the line $\mathcal{T} \cap \mathcal{F}$ (namely $\mathcal{T}$ and $\mathcal{C}(\mathcal{T} \cap \mathcal{F}, L)$) and hence is isomorphic to $\mathbb{G}_3$ by Theorem 4, Lemma 10 and Corollary 2. Let $\mathcal{T}'$ denote the unique $Q(5,2)$-quad of $\mathcal{C}(\mathcal{T}, L)$ through x different from $\mathcal{T}$ and $\mathcal{C}(\mathcal{T} \cap \mathcal{F}, L)$. Then $L \subset \mathcal{T}'$ since $\mathcal{T}' \cap \mathcal{F}$ is a special line.

(b) Let $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ denote three different $Q(5,2)$-quads through x which are not contained in $\mathcal{F}$. By the proof of (a), we know that $\mathcal{T}_1$ and $\mathcal{T}_2$ are contained in a $\mathbb{G}_3$-hex $\mathcal{H}_3$. Hence $\mathcal{T}_1$ and $\mathcal{T}_2$ intersect in a line M_3. In a similar way one can define hexes $\mathcal{H}_1$ and $\mathcal{H}_2$, and lines M_1 and M_2. Now, $\mathcal{H}_1 \cap \mathcal{H}_2 \cap \mathcal{H}_3 = (\mathcal{H}_1 \cap \mathcal{H}_2) \cap (\mathcal{H}_1 \cap \mathcal{H}_3) = \mathcal{T}_3 \cap \mathcal{T}_2 = M_1$. Similarly $M_2 = M_3 = \mathcal{H}_1 \cap \mathcal{H}_2 \cap \mathcal{H}_3$. Hence, all $Q(5,2)$-quads through x not contained in $\mathcal{F}$ have a common line A_x. ■

Corollary 3 *Let* $x \in \mathcal{F}$. *The* $n-1$ $Q(5,2)$*-quads through* A_x *partition the set of lines through* x *which are not contained in* $\mathcal{F} \cup A_x$.

Proof. The $n-1$ $Q(5,2)$-quads through A_x determine $1 + 3(n-1) = 3n-2$ lines through x which are not contained in $\mathcal{F}$. The result now follows since $\delta = 3n-2$. ■

Lemma 21 *For every $x \in \mathcal{F}$, $\mathcal{G}(\mathcal{S}, x)$ is isomorphic to $\mathcal{G}(\mathbb{G}_n)$.*

Proof. Let $\mathcal{F}'$ denote a geodetically closed sub near $2(n-1)$-gon of $\mathbb{G}_n$ isomorphic to $\mathbb{G}_{n-1}$, let $x' \in \mathcal{F}'$ and let $A_{x'}$ denote the unique special line through x' not contained in $\mathcal{F}'$. Since $\mathrm{Aut}(\mathbb{G}_{n-1})$ acts transitively on the set of points of $\mathbb{G}_{n-1}$, there exists an isomorphism ϕ from $\mathcal{F}$ to $\mathcal{F}'$ mapping x to x'. For every line K of $\mathcal{F}$ through x, we define $\theta(K) = \phi(K)$. We will now extend θ in such a way it determines an isomorphism between $\mathcal{L}(\mathcal{S}, x)$ and $\mathcal{L}(\mathbb{G}_n, x')$. Our result then follows from Lemma 7.

Extension of θ. We put $\theta(A_x) = A_{x'}$. Let K and K' denote two arbitrary special lines of $\mathcal{F}$ through x. Let K, A_x, L_1, L_2 and L_3 denote the five lines of $\mathcal{C}(K, A_x)$ through x. Similarly, let K', A_x, L_1', L_2' and L_3' denote the five lines of $\mathcal{C}(K', A_x)$ through x. Let $\theta(L_1)$ be one of the three lines of $\mathcal{C}(\theta(K), A_{x'})$ through x' different from $\theta(K)$ and $A_{x'}$. Now, let M be an arbitrary line through x not contained in $\mathcal{F} \cup \mathcal{C}(K, A_x)$. The quad $\mathcal{C}(L_1, M)$ is a $W(2)$-quad and intersects $\mathcal{F}$ in an ordinary line N. The quad $\mathcal{C}(A_x, M)$ is a $Q(5,2)$-quad and intersects $\mathcal{F}$ in a special line N'. The hex $\mathcal{C}(A_x, L_1, M)$ is isomorphic to $\mathbb{G}_3$ and intersects $\mathcal{F}$ in the $Q(5,2)$-quad $\mathcal{C}(K, N')$. Clearly N is contained in $\mathcal{C}(K, N')$. The hex $\mathcal{C}(A_{x'}, \theta(K), \theta(N'))$ is isomorphic to $\mathbb{G}_3$ and contains the lines $\theta(L_1)$ and $\theta(N)$. The quad $\mathcal{C}(\theta(L_1), \theta(N))$ is isomorphic to $W(2)$ and we put $\theta(M)$ equal to the unique line of $\mathcal{C}(\theta(L_1), \theta(N))$ through x' different from $\theta(L_1)$ and $\theta(M)$. Clearly $\theta(M) \in \mathcal{C}(A_{x'}, \theta(N'))$. We already defined $\theta(L)$ for all lines L through x different from L_2 and L_3. For each $i \in \{2,3\}$, the quad $\mathcal{C}(L_i, L_1')$ is isomorphic to $W(2)$ and intersects $\mathcal{F}$ in a line P. Again $\mathcal{C}(\theta(P), \theta(L_1'))$ is a $W(2)$-quad and we put $\theta(L_i)$ equal to the unique line of $\mathcal{C}(\theta(P), \theta(L_1'))$ through x' different from $\theta(P)$ and $\theta(L_1')$. Clearly, $\theta(L_i) \in \mathcal{C}(A_{x'}, \theta(K))$. One easily sees that θ is a bijection between the set of lines of $\mathcal{S}$ through x and the set of lines of $\mathbb{G}_n$ through x'.

A linear space on a certain set of points is completely determined if all lines of size at least three are know. The linear spaces $\mathcal{L}(\mathcal{S}, x)$ and $\mathcal{L}(\mathbb{G}_n, x')$ each contain $\frac{n(n-1)}{2}$ lines of size 5 and $\frac{3n(n-1)(n-2)}{2}$ lines of size 3. So, in order to prove that θ determines an isomorphism, it suffices to verify that θ maps lines of size $r \in \{3,5\}$ in $\mathcal{L}(\mathcal{S}, x)$ to lines of size r in $\mathcal{L}(\mathbb{G}_n, x')$. By construction (see above), this holds for the lines of size 5. So, let $\delta = \{M_1, M_2, M_3\}$ denote a line of size 3 in $\mathcal{L}(\mathcal{S}, x)$ and let Q_δ denote the $W(2)$-quad corresponding with it. We will now prove that $\{\theta(M_1), \theta(M_2), \theta(M_3)\}$ is a line of size 3 in $\mathcal{L}(\mathbb{G}_n, x')$. This trivially holds if $Q_\delta \subset \mathcal{F}$. Suppose therefore that M_1, M_2 are outside $\mathcal{F}$ and that M_3 is inside $\mathcal{F}$. We may also suppose that $M_1 \neq L_1 \neq M_2$. One of the following cases certainly occurs.

(I) The case $M_1, M_2 \in \{L_2, L_3, L_1', L_2', L_3'\}$.
Let L_1'', L_2'' and L_3'' denote the three lines of $\mathcal{C}(K, K')$ through x different from K and K'. The set $\{L_1, L_2, L_3, L_1', L_2', L_3', L_1'', L_2'', L_3''\}$ together with the subsets $\{L_1, L_2, L_3\}$, $\{L_1', L_2', L_3'\}$, $\{L_1'', L_2'', L_3''\}$, $\{L_i, L_j', \mathcal{C}(L_i, L_j') \cap \mathcal{F}\}$, $i, j \in \{1,2,3\}$, define an affine plane $\mathcal{A}$ of order 3. In a similar way, an affine plane $\mathcal{A}'$ can be defined on the set $\{\theta(L_1), \ldots, \theta(L_3'')\}$. The set $\{\theta(L_1), \ldots, \theta(L_3'')\}$ also carries the structure of an affine plane $\mathcal{A}^\theta$ if one considers all subsets of the form $\{\theta(P_1), \theta(P_2), \theta(P_3)\}$ where $\{P_1, P_2, P_3\}$ is a line of $\mathcal{A}$. Now, $\mathcal{A}'$ and $\mathcal{A}^\theta$ have the following eight lines in common:

$\{\theta(L_1), \theta(L_2), \theta(L_3)\}$, $\{\theta(L'_1), \theta(L'_2), \theta(L'_3)\}$, $\{\theta(L''_1), \theta(L''_2), \theta(L''_3)\}$, $\{\theta(L_1), \theta(L'_1), \mathcal{C}(\theta(L_1), \theta(L'_1)) \cap \mathcal{F}'\}$, $\{\theta(L_1), \theta(L'_2), \mathcal{C}(\theta(L_1), \theta(L'_2)) \cap \mathcal{F}'\}$, $\{\theta(L_1), \theta(L'_3), \mathcal{C}(\theta(L_1), \theta(L'_3)) \cap \mathcal{F}'\}$, $\{\theta(L_2), \theta(L'_1), \mathcal{C}(\theta(L_2), \theta(L'_1)) \cap \mathcal{F}'\}$, $\{\theta(L_3), \theta(L'_1), \mathcal{C}(\theta(L_3), \theta(L'_1)) \cap \mathcal{F}'\}$. Hence $\mathcal{A}' = \mathcal{A}^\theta$. This is precisely what we needed to prove.

(II) The case $\{M_1, M_2\} \cap \{L_1, L_2, L_3\} = \emptyset$.
The quad $\mathcal{C}(A_x, M_i)$, $i \in \{1, 2\}$, intersects $\mathcal{F}$ in a special line P_i. Clearly, $P_1 \neq P_2$. The $W(2)$-quad $\mathcal{C}(L_1, M_i)$, $i \in \{1, 2\}$, intersects $\mathcal{F}$ in an ordinary line N_i which is contained in the $Q(5, 2)$-quad $\mathcal{C}(P_i, K)$. Since N_i is ordinary, $\mathcal{C}(P_i, K)$ is the unique $Q(5, 2)$ quad through N_i. Since $\mathcal{C}(P_1, K) \neq \mathcal{C}(P_2, K)$, $\mathcal{C}(N_1, N_2)$ is not a $Q(5, 2)$-quad. The hex $\mathcal{H} = \mathcal{C}(L_1, M_1, M_2)$ intersects $\mathcal{F}$ in the quad $\mathcal{C}(N_1, N_2)$. The line M_3 belongs to $\mathcal{C}(N_1, N_2)$ and is different from N_1 and N_2. Hence $\mathcal{C}(N_1, N_2) \cong W(2)$. Since also $\mathcal{C}(\theta(N_1), \theta(N_2)) \cong W(2)$, the lines $\theta(N_1)$, $\theta(N_2)$ and $\theta(M_3)$ are precisely the three lines of $\mathcal{C}(\theta(N_1), \theta(N_2))$ through x'. Since $\mathcal{C}(\theta(L_1), \theta(M_1)) \cap \mathcal{F}' = \theta(N_1)$, $\mathcal{C}(\theta(L_1), \theta(M_2)) \cap \mathcal{F}' = \theta(N_2)$ and $\mathcal{C}(\theta(L_1), \theta(M_1), \theta(M_2)) \cap \mathcal{F} = \mathcal{C}(\theta(N_1), \theta(N_2))$, we necessarily have that $\mathcal{C}(\theta(M_1), \theta(M_2)) \cap \mathcal{F}' = \theta(M_3)$. This is precisely what we needed to prove.

(III) The case $\{M_1, M_2\} \cap \{L'_1, L'_2, L'_3\} = \emptyset$.
By (I) and (II), θ maps the lines $\{L'_1, M_1, \mathcal{C}(L'_1, M_1) \cap \mathcal{F}\}$ and $\{L'_1, M_2, \mathcal{C}(L'_1, M_2) \cap \mathcal{F}\}$ of $\mathcal{L}(\mathcal{S}, x)$ to lines of $\mathcal{L}(\mathbb{G}_n, x')$. With a similar reasoning as in (II), we then derive that also $\{M_1, M_2, \mathcal{C}(M_1, M_2) \cap \mathcal{F}\}$ is mapped to a line of $\mathcal{L}(\mathbb{G}_n, x')$. ∎

Lemma 22 *Every point y of $\mathcal{S}$ is contained in a big geodetically closed sub near polygon isomorphic to $\mathbb{G}_{n-1}$. Hence $\mathcal{G}(\mathcal{S}, y) \cong \mathcal{G}(\mathbb{G}_n)$.*

Proof. We may suppose that $y \notin F$, then y is collinear with a unique point $\pi(y)$ of F. Call a line L through $\pi(y)$ special if it is not contained in a $W(2)$-quad and ordinary otherwise. Since $\mathcal{G}(\mathcal{S}, \pi(y)) \cong \mathcal{G}(\mathbb{G}_n)$, there are precisely n special lines $L_1, \ldots, L_n$ through $\pi(y)$. We may suppose that $y\,\pi(y) \subset \mathcal{C}(L_1, L_2)$. For every $i \in \{2, \ldots, n\}$, we put $\mathcal{F}_i := \mathcal{C}(L_1, \ldots, L_i)$. Since $\mathcal{G}(\mathcal{S}, \pi(y)) \cong \mathcal{G}(\mathbb{G}_n)$, we have the following for every $i \in \{2, \ldots, n-1\}$:

- (i) $\mathcal{F}_i$ is a dense geodetically closed sub near polygon of order $(2, \frac{3i^2-3i-2}{2})$;
- (ii) every quad of $\mathcal{F}_{i+1}$ through $\pi(y)$ either is contained in $\mathcal{F}_i$ or intersects $\mathcal{F}_i$ in a line.

By (i) and Theorem 4, $\mathcal{F}_2 \cong Q(5, 2)$ and $\mathcal{F}_3 \cong \mathbb{G}_3$. Suppose now that $\mathcal{F}_i \cong \mathbb{G}_i$ for a certain $i \in \{3, n-2\}$. By (ii) and Lemma 6, $\mathcal{F}_i$ is big in $\mathcal{F}_{i+1}$. By our Main Theorem (recall that our proof is by induction) it then follows that $\mathcal{F}_{i+1}$ is isomorphic to either $\mathbb{G}_{i+1}$, $\mathbb{G}_i \otimes \mathbb{G}_2$ or $\mathbb{G}_i \times L$. By (i), we have $\mathcal{F}_{i+1} \cong \mathbb{G}_{i+1}$. Now, $y \in \mathcal{F}_{n-1}$ and $\mathcal{F}_{n-1} \cong \mathbb{G}_{n-1}$ is big in $\mathcal{S}$. By Lemma 21 applied to $\mathcal{F}_{n-1}$ instead of $\mathcal{F}$, $\mathcal{G}(\mathcal{S}, y) \cong \mathcal{G}(\mathbb{G}_n)$. ∎

Call a line L of $\mathcal{S}$ *special* if it is not contained in a $W(2)$-quad, and *ordinary* otherwise. Since $\mathcal{G}(\mathcal{S}, y) \cong \mathcal{G}(\mathbb{G}_n)$ for every point y of $\mathcal{S}$, every point of $\mathcal{S}$ is incident with n special lines and $\frac{3}{2}n(n-1)$ ordinary lines. Let V_k, $k \in \{1, \ldots, n\}$, denote the set of all geodetically closed sub near $2k$-gons generated by k special lines through a fixed point. If $\mathcal{F} \in V_k$, $k \in \{1, \ldots, n-1\}$, then a similar reasoning as in the proof of Lemma 22 gives that $\mathcal{F} \cong \mathbb{G}_k$. Together with Corollary 2 this implies that every element of V_k, $k \in \{1, \ldots, n\}$, has $m_k := \frac{3^k \cdot (2k)!}{2^k \cdot k!}$ points.

Lemma 23 *A subgrid* G_1 *of* $\mathcal{Q} \cong Q(5,2)$ *defines a unique partition* $\{G_1, G_2, G_3\}$ *of* $\mathcal{Q}$ *into three subgrids.*

Proof. For a point x of $\mathcal{Q}$, let $x^\perp$ denote the set of points of $\mathcal{Q}$ collinear with x. Call two vertices $x, y \in \mathcal{Q} \setminus G_1$ equivalent if $x^\perp \cap G_1$ and $y^\perp \cap G_1$ are equal or disjoint. There are two equivalence classes C_2 and C_3 each containing 9 points. A point $x \in C_i$ is contained in three lines meeting G_1 and two lines which are entirely contained in C_i. So, each C_i contains $\frac{9 \cdot 2}{3} = 6$ lines. Clearly, a grid G_i is formed by the 9 points and 6 lines in C_i. The uniqueness of $\{G_1, G_2, G_3\}$ is also obvious. ∎

Lemma 24 *Let* M_1*,* M_2 *and* M_3 *be three mutual disjoint lines in a subgrid* G *of* $\mathcal{S}$*. If* M_1 *and* M_2 *are special, then also* M_3 *is special.*

Proof. There exists an element $\mathcal{F} \in V_{n-1}$ through M_2 not containing G. Since $\mathcal{R}_\mathcal{F} \in \mathrm{Aut}(\mathcal{S})$, $M_3 = \mathcal{R}_\mathcal{F}(M_1)$ is special. ∎

Lemma 25 *Every* $Q(5,2)$*-quad* $\mathcal{Q}$ *of* $\mathcal{S}$ *can be partitioned into three grids, such that a line of* $\mathcal{Q}$ *is special if and only if it is contained in one of these grids.*

Proof. If $x \in \mathcal{Q}$, then $\mathcal{G}(\mathcal{S}, x) \cong \mathbb{G}_n$ and hence exactly two from the five lines of $\mathcal{Q} \cong Q(5,2)$ through x are special. Since $\mathcal{Q}$ contains 27 points, it has exactly $\frac{27 \cdot 2}{3} = 18$ special lines. Consider a special line $L \subseteq \mathcal{Q}$ and let M_1, M_2 and M_3 denote the three special lines of $\mathcal{Q}$ intersecting L in a point. By Lemma 24, M_1, M_2 and M_3 are contained in a grid G_1. Let G_2 and G_3 denote the subgrids of $\mathcal{Q}$ as in Lemma 23. At most 10 from the 18 special lines meet G_1; hence $G_2 \cup G_3$ contains two intersecting special lines N_1 and N_2. We may suppose that $N_1, N_2 \subseteq G_3$. For every line P of G_2, there exists a unique $i \in \{1,2,3\}$ and a unique $j \in \{1,2\}$ such that P, M_i and N_j are contained in a grid. Hence by Lemma 24, every line of G_2 is special. Since $\mathcal{Q}$ contains exactly 12 special lines disjoint from G_2, all lines of G_1 and G_3 are special. This proves our lemma. ∎

Define the following relation R on the set $V := V_{n-1}$. For two elements $v_1, v_2 \in V$, we say that $(v_1, v_2) \in R$ if exactly one of the following holds:

(i) $v_1 = v_2$

(ii) $v_1 \cap v_2 = \emptyset$ and every line meeting v_1 and v_2 is special.

Lemma 26 *The relation R is an equivalence relation and every equivalence class contains exactly 3 elements.*

Proof. Let $v \in V$ be arbitrary. Every point $a \in v$ is contained in a unique special line $L_a = \{a, a_1, a_2\}$ not contained in v, and we define $\Omega_a := \{v_{a_1}, v_{a_2}\}$ where v_{a_i} denotes the unique element of V through a_i not containing L_a. It suffices to prove that $\Omega_a = \Omega_b$ for all $a, b \in v$.

Suppose first that $\mathrm{d}(a, b) = 1$. Let c denote the unique third point on the line ab and let v' denote an element of V through c not containing ab. Since $\mathcal{R}_{v'} \in \mathrm{Aut}(\mathcal{S})$, $\mathcal{R}_{v'}(L_a)$ is a special line through b and hence equal to L_b. As a consequence L_b is contained in the quad $\mathcal{Q} := \mathcal{C}(b, L_a)$. Since L_a is special, $\mathcal{Q}$ is not isomorphic to $W(2)$. Suppose that $\mathcal{Q}$ is a grid. Since v_{a_i} is big, $\mathcal{Q} \cap v_{a_i}$ is a line that meets L_b. Since $L_b \cap v_{a_i} \neq \emptyset$, $i \in \{1, 2\}$, $\Omega_b = \{v_{a_1}, v_{a_2}\} = \Omega_a$. Suppose that $\mathcal{Q}$ is a $\mathcal{Q}(5, 2)$-quad. Since $\mathcal{Q} \in V_2$ and $v, v_{a_1}, v_{a_2} \in V$, $\mathcal{Q} \cap v$, $\mathcal{Q} \cap v_{a_1}$ and $\mathcal{Q} \cap v_{a_2}$ are special lines (see Lemma 9). By Lemma 25, the unique line through b intersecting $\mathcal{Q} \cap v_{a_i}$ is special and hence equal to L_b. Since $L_b \cap v_{a_i} \neq \emptyset$, $i \in \{1, 2\}$, $\Omega_b = \{v_{a_1}, v_{a_2}\} = \Omega_a$.

If a and b are not collinear, consider then a path $a = c_0, \ldots, c_k = b$ of length $k = d(a, b)$ between a and b. Then $\Omega_a = \Omega_{c_0} = \cdots = \Omega_{c_k} = \Omega_b$. ■

Lemma 27 *Let v_1, v_2 and v_3 be three different elements of V for which $(v_1, v_2) \in R$. Then $v_1 \cap v_3 \neq \emptyset$ if and only if $v_2 \cap v_3 \neq \emptyset$.*

Proof. If $a \in v_1 \cap v_3$, then v_3 necessarily contains the unique special line L_a through a not contained in v_1. Since $L_a \cap v_2 \neq \emptyset$, the lemma follows. ■

Lemma 28 *Let $v_1, v_2, v_3, v_4 \in V$ such that $(v_i, v_j) \notin R$ for all $i, j \in \{1, 2, 3, 4\}$ with $i \neq j$. If $v_1 \cap v_2 = \emptyset$ and $v_3 = \mathcal{R}_{v_2}(v_1)$, then v_4 intersects at least one of v_1, v_2 and v_3.*

Proof. Since every point of $\mathcal{S}$ is contained in n elements of V, we have $|V| = \frac{m_n \cdot n}{m_{n-1}} = 3n(2n - 1)$.

(i) Let N_1 denote the number of elements of V intersecting v_1, v_2 and v_3. Every line intersecting v_1 and v_2 is ordinary and hence is contained in $n-2$ elements of V. Each of these $n - 2$ elements intersects v_1 in an element of V_{n-2}. Hence $N_1 = \frac{m_{n-1} \cdot (n-2)}{m_{n-2}} = 3(n - 2)(2n - 3)$.

(ii) Let N_2 denote the number of elements of $V \setminus \{v_1\}$ meeting v_1 and disjoint from v_2 and v_3. By (i), every point of v_1 is contained in $n - 2$ elements of V which intersect v_2 and v_3. Hence every point of v_1 is contained in a unique element of $V \setminus \{v_1\}$ disjoint from v_2 and v_3. This element intersects v_1 in an element of V_{n-2}. Hence $N_2 = \frac{m_{n-1}}{m_{n-2}} = 3(2n - 3)$.

(iii) There are $N_3 = 9$ elements of V belonging to one of the equivalence classes determined by v_1, v_2 and v_3.

The lemma now follows since $N_1 + 3N_2 + N_3 = |V|$. ■

Let Γ be the graph whose vertices are the equivalence classes determined by R with two classes γ_1 and γ_2 adjacent if and only if $v_1 \cap v_2 = \emptyset$ for every $v_1 \in \gamma_1$ and every $v_2 \in \gamma_2$. The graph Γ has $\frac{|V|}{3} = \binom{2n}{2}$ vertices.

Lemma 29 *The graph Γ is regular with valency $k(\Gamma) = 4(n-1)$.*

Proof. Let v be a fixed element of V. From the $3n(2n-1)$ elements in V, 3 are contained in the equivalence class of v, and $\frac{m_{n-1}\cdot(n-1)}{m_{n-2}} = 3(n-1)(2n-3)$ intersect v in an element of V_{n-2}. By Lemma 27 it then follows that $k(\Gamma) = \frac{3n(2n-1)-3(n-1)(2n-3)-3}{3} = 4(n-1)$. ∎

Lemma 30 *Every 2 adjacent vertices γ_1 and γ_2 of Γ are contained in two maximal cliques, one of size 3 and one of size $2n-1$.*

Proof. Let $v_1 \in \gamma_1$, $v_2 \in \gamma_2$, let v_3 denote the reflection of v_2 about v_1 and let γ_3 denote the equivalence class of v_3. By Lemma 28, $\{\gamma_1, \gamma_2, \gamma_3\}$ is a maximal clique. Let $C \neq \{\gamma_1, \gamma_2, \gamma_3\}$ denote another maximal clique through γ_1 and γ_2. If $\gamma_4 \in C \setminus \{\gamma_1, \gamma_2\}$, then every $v_4 \in \gamma_4$ intersects v_3. By the proof of Lemma 28, there are $N_2 = 3(2n-3)$ mutually disjoint elements in $V \setminus \{v_3\}$ which intersect v_3 and are disjoint from $v_1 \cup v_2$. By Lemma 27, these elements of V correspond to $\frac{N_2}{3} = 2n-3$ vertices of Γ. The maximal clique C necessarily consists of γ_1, γ_2 and these $2n-3$ vertices of Γ. This proves our lemma. ∎

Lemma 31 *There is a bijective correspondence between the maximal cliques of size $2n-1$ in Γ and the elements of $B = \{\bar{e}_0, \ldots, \bar{e}_{2n-1}\}$. There is a bijective correspondence between the vertices of Γ and the pairs of the set B.*

Proof. The graph Γ has $\frac{|\Gamma|\cdot k(\Gamma)}{(2n-1)\cdot(2n-2)} = 2n$ maximal cliques of size $2n-1$, proving the first part of the lemma. Since every vertex of Γ is contained in $\frac{k(\Gamma)}{2n-2} = 2$ maximal cliques, it corresponds with a subset of size 2 of B. By Lemma 30, every pair of B corresponds to at most one vertex of Γ. The second part of the lemma now follows since there are as many vertices in Γ as there are pairs in B.

Lemma 32 *Let v_1, v_2 denote two nonequivalent disjoint elements of V, let v_3 denote the reflection of v_2 around v_1, and let γ_k, $k \in \{1,2,3\}$, denote the equivalence class determined by v_k. Then there exist $\bar{f}_1, \bar{f}_2, \bar{f}_3 \in B$ such that γ_j, $j \in \{1,2,3\}$, corresponds to $\{\bar{f}_j, \bar{f}_{j+1}\}$, where indices are taken modulo 3.*

Proof. Let γ_1 correspond to $\{\bar{f}_1, \bar{f}_2\} \subseteq B$, γ_2 to $\{\bar{g}_1, \bar{g}_2\} \subseteq B$ and γ_3 to $\{\bar{h}_1, \bar{h}_2\} \subseteq B$. Since γ_1, γ_2 and γ_3 are not contained in a maximal clique of size $2n-1$, $\{\bar{f}_1, \bar{f}_2\} \cap \{\bar{g}_1, \bar{g}_2\} \cap \{\bar{h}_1, \bar{h}_2\} = \emptyset$. Since there is a unique maximal clique of size $2n-1$ through γ_1 and γ_2, $|\{\bar{f}_1, \bar{f}_2\} \cap \{\bar{g}_1, \bar{g}_2\}| = 1$. Similarly, $|\{\bar{f}_1, \bar{f}_2\} \cap \{\bar{h}_1, \bar{h}_2\}| = 1$ and $|\{\bar{g}_1, \bar{g}_2\} \cap \{\bar{h}_1, \bar{h}_2\}| = 1$. The lemma now immediately follows. ∎

We define X as the set of all points of weight 2 in $PG(2n-1, 4)$ with respect to a fixed reference system.

Lemma 33 *The point-line geometry* Δ *with point set* V *and line set* $\{\{v_1, v_2, \mathcal{R}_{v_2}(v_1)\} | v_1, v_2 \in V, v_1 \cap v_2 = \emptyset\}$ *is isomorphic to the point-line geometry* Δ' *whose points are the elements of* X *and whose lines are those lines* L *of* $\mathrm{PG}(2n-1, 4)$ *for which* $|L \cap X| = 3$ *(natural incidence).*

Proof. We first construct a bijection between V and X. For every $i \in \{1, \ldots, 2n-1\}$, the equivalence class corresponding to $\{\bar{e}_0, \bar{e}_i\}$ contains three elements of V which can labeled with the three elements of the set $\{\langle \bar{e}_0 + \alpha \bar{e}_i \rangle | \alpha \in \mathrm{GF}(4)^*\} \subseteq X$. For all $i, j \in \{1, 2, \ldots, 2n-1\}$ with $i < j$ and every $\alpha \in GF(4)^*$, the reflection of $\langle \bar{e}_0 + \alpha \bar{e}_j \rangle$ (regarded as element of V) around $\langle \bar{e}_0 + \bar{e}_i \rangle$ is labeled with the element $\langle \bar{e}_i + \alpha \bar{e}_j \rangle$ of X. In this way, we have a bijection between V and X.

For all $i, j \in \{1, 2, \ldots, 2n-1\}$ with $i < j$, we now define a binary operation $\otimes_{ij}$ on $\mathrm{GF}(4)^*$ in the following way: $\langle \bar{e}_i + (\alpha \otimes_{ij} \beta) \bar{e}_j \rangle$ is the reflection of $\langle \bar{e}_0 + \beta \bar{e}_j \rangle$ about $\langle \bar{e}_0 + \alpha \bar{e}_i \rangle$. Clearly $\otimes_{ij}$ determines a latin square of order 3 on the set $\mathrm{GF}(4)^*$. Since $1 \otimes_{ij} \alpha = \alpha$ for every $\alpha \in GF(4)^*$, we necessarily have $\alpha \otimes_{ij} \beta = \alpha^{\epsilon_{ij}} \cdot \beta$ for some $\epsilon_{ij} \in \{+1, -1\}$.

Let $i, j, k \in \{1, \ldots, 2n-1\}$ such that $i < j < k$ and let $\alpha, \beta, \gamma \in \mathrm{GF}(4)^*$. Put $v = \langle \bar{e}_0 + \gamma \bar{e}_i \rangle$, $v_1 = \langle \bar{e}_0 + \alpha \bar{e}_j \rangle$, $v_2 = \langle \bar{e}_0 + \beta \bar{e}_k \rangle$ and $v_3 = \langle \bar{e}_j + (\alpha^{\epsilon_{jk}} \cdot \beta) \bar{e}_k \rangle$. Since $v_3 = \mathcal{R}_{v_1}(v_2)$ and $\mathcal{R}_v \in \mathrm{Aut}(\mathcal{S})$, the reflection of $\mathcal{R}_v(v_2)$ around $\mathcal{R}_v(v_1)$ equals $\mathcal{R}_v(v_3)$. Hence, the reflection of $\langle \bar{e}_i + (\gamma^{\epsilon_{ij}} \cdot \alpha) \bar{e}_j \rangle$ around $\langle \bar{e}_i + (\gamma^{\epsilon_{ik}} \cdot \beta) \bar{e}_k \rangle$ equals $\langle \bar{e}_j + (\alpha^{\epsilon_{jk}} \cdot \beta) \bar{e}_k \rangle$. In particular, the reflection of $\langle \bar{e}_i + \alpha \bar{e}_j \rangle$ around $\langle \bar{e}_i + \beta \bar{e}_k \rangle$ equals $\langle \bar{e}_j + (\alpha^{\epsilon_{jk}} \cdot \beta) \bar{e}_k \rangle$. Hence $(\gamma^{\epsilon_{ij}} \cdot \alpha)^{\epsilon_{jk}} \cdot (\gamma^{\epsilon_{ik}} \cdot \beta) = (\alpha^{\epsilon_{jk}} \cdot \beta)$ or $\epsilon_{ij} \epsilon_{jk} = -\epsilon_{ik}$. Putting $\epsilon_{11} = -1$, we have that $\epsilon_{1j} \epsilon_{jk} = -\epsilon_{1k}$ for all $j, k \in \{1, \ldots, 2n-1\}$ with $j < k$.

For a point $v \in V$ with label $\langle \bar{e}_i + \alpha \bar{e}_j \rangle$, $i < j$, we put $\theta(v) := \langle \bar{e}_i + \alpha^{\epsilon_{1j}} \bar{e}_j \rangle$. Clearly θ is a bijection between V and X. Now, choose i, j and k such that $0 \leq i < j < k \leq 2n-1$, and let $\alpha, \beta \in \mathrm{GF}(4)^*$. Since $v_1 := \theta^{-1}(\langle \bar{e}_i + \alpha \bar{e}_j \rangle)$ and $v_2 := \theta^{-1}(\langle \bar{e}_i + \beta \bar{e}_k \rangle)$ have respective labels $\langle \bar{e}_i + \alpha^{\epsilon_{1j}} \bar{e}_j \rangle$ and $\langle \bar{e}_i + \beta^{\epsilon_{1k}} \bar{e}_k \rangle$, the reflection v_3 of v_2 around v_1 has label $\langle \bar{e}_j + (\alpha^{\epsilon_{1j} \epsilon_{jk}} \beta^{\epsilon_{1k}}) \bar{e}_k \rangle$. Hence $\theta(v_3) = \langle \bar{e}_j + (\alpha^{\epsilon_{1j} \epsilon_{jk} \epsilon_{1k}} \beta^{\epsilon_{1k} \epsilon_{1k}}) \bar{e}_k \rangle = \langle \bar{e}_j + (\alpha^{-1} \beta) \bar{e}_k \rangle$. It is now easily seen that θ is an isomorphism between Δ and Δ'. ■

Recall that $\mathbb{G}_n = (Y, Y', \mathrm{I})$, where Y is the set of all good subspaces of dimension $n-1$ and where Y' is the set of all good subspaces of dimension $n-2$. We take the following facts from [7]: (a) if $\pi \in Y$, then $\mathcal{G}_\pi$ consists of n elements of X, (b) if $\pi \in Y'$ is a special line of $\mathbb{G}_n$, then $\mathcal{G}_\pi$ consists of $n-1$ elements of X, (c) if $\pi \in Y'$ is an ordinary line of $\mathbb{G}_n$, then $\mathcal{G}_\pi$ consists of $n-2$ elements of X and one point of weight 4.

Every point x of $\mathcal{S}$ is contained in n elements $v_1, \ldots, v_n$ of V. Since $v_i \cap v_j \neq \emptyset$, the supports of $\theta(v_i)$ and $\theta(v_j)$ are disjoint. We define $\phi(x) := \langle \theta(v_1), \ldots, \theta(v_n) \rangle$. Clearly $\phi(x) \in Y$.

Lemma 34 *The map* $\phi : \mathcal{P} \mapsto Y$ *is bijective.*

Proof. Let $\pi \in Y$, then $\{v_1, \ldots, v_n\} := \theta^{-1}(X \cap \pi)$ is a set of n elements of V and $v_1 \cap \cdots \cap v_n$ is a geodetically closed sub near polygon. Since a line of $\mathcal{S}$ is contained in at most $n-1$ elements of V, $|v_1 \cap \cdots \cap v_n| \leq 1$. If $\pi = \phi(x)$, then $\{x\} = v_1 \cap \cdots \cap v_n$, proving that ϕ is injective. Since $|Y| = |\mathcal{P}| = \frac{3^n \cdot (2n)!}{2^n \cdot n!}$, ϕ necessarily is bijective. ■

For a line $L = \{x_1, x_2, x_3\}$ of $\mathcal{S}$, we put $\phi'(L) = \phi(x_1) \cap \phi(x_2) \cap \phi(x_3)$.

Lemma 35 *For every line L, $\phi'(L) \in Y'$.*

Proof. (A) Suppose that L is special. Let $v_1, \ldots, v_{n-1}$ denote the $n-1$ elements of V through L, and let w_i, $i \in \{1, 2, 3\}$, denote the unique element of V through x_i not containing L. Clearly $\phi'(L) = \langle \theta(v_1), \ldots, \theta(v_{n-1}) \rangle \in Y'$.
(B) Suppose that L is an ordinary line. Let $v_1, \ldots, v_{n-2}$ denote those elements of V through L, and let u_i and w_i denote the two elements of V through x_i not containing L. We may suppose that $u_3 = \mathcal{R}_{u_1}(u_2)$. Then $w_2 = \mathcal{R}_{w_1}(u_3)$ and $w_3 = \mathcal{R}_{u_1}(w_2)$. Putting $\theta(u_1) = \langle \bar{e}_0 + \alpha \bar{e}_1 \rangle$, $\theta(w_1) = \langle \bar{e}_2 + \beta \bar{e}_3 \rangle$ and $\theta(u_2) = \langle \bar{e}_1 + \gamma \bar{e}_2 \rangle$, we find $\theta(u_3) = \langle \bar{e}_0 + \alpha\gamma \bar{e}_2 \rangle$, $\theta(w_2) = \langle \bar{e}_0 + \alpha\beta\gamma \bar{e}_3 \rangle$ and $\theta(w_3) = \langle \bar{e}_1 + \beta\gamma \bar{e}_3 \rangle$. One easily calculates that $\phi'(L) = \langle \theta(v_1), \ldots, \theta(v_{n-2}), \langle \bar{e}_0 + \alpha \bar{e}_1 + \alpha\gamma \bar{e}_2 + \alpha\beta\gamma \bar{e}_3 \rangle \rangle \in Y'$. ■

Lemma 36 *The map $\phi' : \mathcal{L} \mapsto Y'$ is bijective.*

Proof. Let $\pi' \in Y'$. If $\pi' = \phi'(L)$, then necessarily $L = \{\phi^{-1}(\pi) | \pi \in Y \text{ and } \pi' \subset \pi\}$. Hence ϕ is injective. Since $|\mathcal{L}| = |Y'| = \frac{3^{n-1}(2n)!(3n-1)}{2^{n+1}(n-1)!}$, ϕ' is bijective. ■

Now, a point x and a line L of $\mathcal{S}$ are incident if and only if $\phi(x)$ and $\phi'(L)$ are incident in $\mathbb{G}_n$. This proves that $\mathcal{S} \cong \mathbb{G}_n$.

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