On Global Solvability and Asymptotic Behavior of a Nonlinear Coupled System with Memory Condition at the Boundary

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Abstract

We consider a nonlinear coupled system of two wave equations with memory condition at the boundary and we study the asymptotic behavior of the corresponding solutions. We proved that the energy decay with the same rate of decay of the relaxation functions, that is, the energy decays exponentially when the relaxation functions decay exponentially and decay polynomially when the relaxation functions decay polynomially.

1. Introduction

The main purpose of this work is to study the asymptotic behavior of the solutions of a nonlinear coupled system of two wave equations with boundary conditions of memory type. To formalize this problem let us take Ω an open bounded set of $\mathbb{R}^n$ with smooth boundary Γ and let us assume that Γ can be divided in two not null parts

$$\Gamma = \Gamma_0 \cup \Gamma_1 \quad \text{with} \quad \bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset.$$

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Let us denote by $\nu(x)$ the unit normal vector at $x \in \Gamma$ outside of Ω and let us consider the following initial boundary value problem

$$u_{tt} - \beta_1 \Delta u + f(u-v) = 0 \quad \text{in} \quad \Omega \times (0,\infty), \tag{1..1}$$
$$v_{tt} - \beta_2 \Delta v - f(u-v) = 0 \quad \text{in} \quad \Omega \times (0,\infty), \tag{1..2}$$

$$u = v = 0 \quad \text{on} \quad \Gamma_0 \times (0,\infty), \tag{1..3}$$
$$u + \int_0^t g_1(t-s)\frac{\partial u}{\partial \nu}(s)ds = 0 \quad \text{on} \quad \Gamma_1 \times (0,\infty), \tag{1..4}$$
$$v + \int_0^t g_2(t-s)\frac{\partial v}{\partial \nu}(s)ds = 0 \quad \text{on} \quad \Gamma_1 \times (0,\infty), \tag{1..5}$$

$$(u(0,x), v(0,x)) = (u_0(x), v_0(x)), \quad (u_t(0,x), v_t(0,x)) = (u_1(x), v_1(x)) \quad \text{in} \quad \Omega. \tag{1..6}$$

Here, u and v are the transverse displacements. The elasticity coefficients β_i are positive, the relaxation functions g_i are positive and non decreasing and the function $f \in C^1(\mathbb{R})$ satisfies

$$f(s)s \geq 0 \quad \forall s \in \mathbb{R}, \quad \forall i = 1,2.$$

Additionally, we suppose that f is superlinear, that is

$$f(s)s \geq (2+\delta)F(s), \quad F(z) := \int_0^z f(s)ds \quad \forall s \in \mathbb{R}, \quad \forall i = 1,2,$$

for some $\delta > 0$ with the following growth conditions

$$|f(x) - f(y)| \leq c(1 + |x|^{\rho-1} + |y|^{\rho-1})|x-y|, \quad \forall x,y \in \mathbb{R}, \quad \forall i = 1,2,$$

for some $c > 0$ and $\rho \geq 1$ such that $(n-2)\rho \leq n$. The integral equations (1..4)-(1..5) describe the memory effects which can be caused, for example, by the interaction with another viscoelastic element. Also, we shall assume that there exists $x_0 \in \mathbb{R}^n$ such that

$$\Gamma_0 = \{x \in \Gamma \ : \ \nu(x)\cdot(x-x_0) \leq 0\},$$
$$\Gamma_1 = \{x \in \Gamma \ : \ \nu(x)\cdot(x-x_0) > 0\}.$$

An example of a set Ω satisfying those properties is showed in Figure 1.

Let us denote $m(x) = x - x_0$. Note that the compactness of Γ_1 implies that there exists a small positive constant δ_0 such that

$$0 < \delta_0 \leq m(x)\cdot\nu(x), \quad \forall x \in \Gamma_1.$$

Dissipative coupled systems of the wave equations with $f(s) = \alpha s$ were studied by several authors, see for example [1, 2, 3, 4] among others. In [4], Komornik and Rao studied a linear system of two compactly coupled wave equations with boundary frictional damping in both equations. They showed the existence, regularity and stability of the corresponding solutions. The stability results obtained in [4] were extended by Aassila [1] for a coupled system with weak frictional damping at infinity. In a second work, Aassila [2], removes the dissipation of one equation and shows

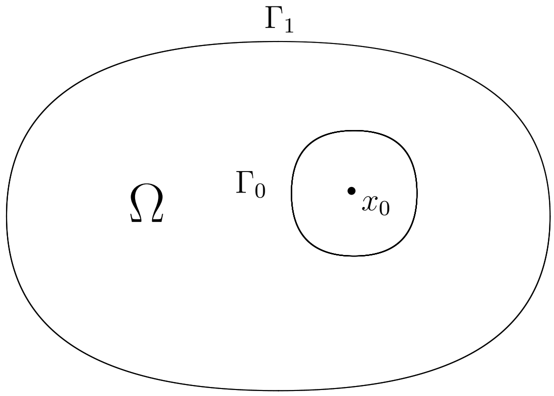

Figure 1. The set Ω.

strong asymptotic stability and non uniform stability for some particular cases depending on the coupling constant. Another similar coupled system with boundary frictional damping on only one of the equations was studied by Alabau [3]. She shows the polynomial decay of the corresponding strong solutions when the speed of wave propagation of the both equations are the same ($\beta_1 = \beta_2$). Concerning to the memory condition at the boundary and $f(s) = \alpha s$ we can cite the work of Oquendo and Santos [6]. They showed the existence, regularity and stability exponential and polynomial of the corresponding solutions. It seems to us that there is no result concerning the asymptotic stability of solutions for the system (1..1)-(1..6) when the system is coupled non linearly. So to fill this gap we study this topic here. The main result of this paper is to show that the solutions of the system (1..1)-(1..6) decays uniformly in time with the same rate of decay of the relaxation functions. More precisely, we show that the solution decays exponentially to zero provided $g_1,\ g_2$ decays exponentially to zero. When the relaxation functions $g_1,\ g_2$ decays polynomially, we show that the corresponding solution also decays polynomially to zero. The method used is based on the construction of a suitable Lyapunov functional $\mathcal{L}$ satisfying

$$\frac{d}{dt}\mathcal{L}(t) \leq -c_1\mathcal{L}(t) + c_2 e^{-\gamma t} \quad \text{ or } \quad \frac{d}{dt}\mathcal{L}(t) \leq -c_1\mathcal{L}(t)^{1+\frac{1}{\alpha}} + \frac{c_2}{(1+t)^{\alpha+1}},$$

for some positive constants c_1, c_2, γ and α. Note that, because of condition (1..3) the solution of the system (1..1)-(1..6) must belong to the following space

$$V := \{v \in H^1(\Omega) \ : \ v = 0 \quad \text{on} \quad \Gamma_0\}.$$

The notations we use in this paper are standard and can be found in Lions' book [5]. In the sequel, by c (sometime $c_1, c_2, \dots$), we denote various positive constants which do neither depend on t nor on the initial data. The remainder of this paper is organized as follows. In section 2 we establish the existence and uniqueness of strong solutions for the system(1..1)-(1..6). In section 3 we prove the exponential decay and in section 4, the polynomial decay. Finally in section 5 will make a final comment on three problems related with the problem in this subject, that is, (1..1)-(1..6), for which the method explored in this paper can be used to solve.

2. Existence of Solutions

In this section we shall study the existence and regularity of solutions for the coupled system (1..1)-(1..6). First, we shall use equations (1..4)-(1..5) to estimate the terms $\frac{\partial u}{\partial \nu}$ and $\frac{\partial v}{\partial \nu}$ on Γ_1. Denoting by

$$(g * \varphi)(t) = \int_0^t g(t-s)\varphi(s)ds,$$

the convolution product operator and differentiating the equations (1..4) and (1..5) we arrive to the following Volterra equations

$$\begin{aligned} \frac{\partial u}{\partial \nu} + \frac{1}{g_1(0)} g_1' * \frac{\partial u}{\partial \nu} &= -\frac{1}{g_1(0)} u_t, \\ \frac{\partial v}{\partial \nu} + \frac{1}{g_2(0)} g_2' * \frac{\partial v}{\partial \nu} &= -\frac{1}{g_2(0)} v_t. \end{aligned}$$

Applying the Volterra's inverse operator, we get

$$\begin{aligned} \frac{\partial u}{\partial \nu} &= -\frac{1}{g_1(0)} \{u_t + k_1 * u_t\}, \\ \frac{\partial v}{\partial \nu} &= -\frac{1}{g_2(0)} \{v_t + k_2 * v_t\}, \end{aligned}$$

where the resolvent kernels satisfy

$$k_i + \frac{1}{g_i(0)} g_i' * k_i = -\frac{1}{g_i(0)} g_i' \quad \text{for} \quad i = 1, 2.$$

Denoting by $\tau_1 = \frac{1}{g_1(0)}$ and $\tau_2 = \frac{1}{g_2(0)}$ the normal derivatives of u and v can be written as

$$\frac{\partial u}{\partial \nu} = -\tau_1 \{u_t + k_1(0)u - k_1(t)u_0 + k_1' * u\} \tag{2..1}$$

$$\frac{\partial v}{\partial \nu} = -\tau_2 \{v_t + k_2(0)v - k_2(t)v_0 + k_2' * v\}. \tag{2..2}$$

Reciprocally, taking initial data such that $u_0 = v_0 = 0$ on Γ_1, the identities (2..1)-(2..2) imply (1..4)-(1..5). Since we are interested in relaxation functions of exponential or polynomial type and since the identities (2..1)-(2..2) involve the resolvent kernels k_i, we want to know if k_i has the same properties. The following lemma answers this question. Let h be a relaxation function and k its resolvent kernel, that is

$$k(t) - k * h(t) = h(t).$$

Lemma 2..1. *If h is a positive continuous function, then k is also a positive continuous function. Moreover,*

1. *If there exist positive constants* c_0 *and* γ *with* $c_0 < \gamma$ *such that*

$$h(t) \leq c_0 e^{-\gamma t},$$

then, the function k *satisfies*

$$k(t) \leq \frac{c_0(\gamma - \epsilon)}{\gamma - \epsilon - c_0} e^{-\epsilon t},$$

for all $0 < \epsilon < \gamma - c_0$.

2. *Given* $p > 1$, *let us denote by* $c_p := \sup_{t \in \mathbb{R}^+} \int_0^t (1+t)^p (1+t-s)^{-p} (1+s)^{-p}\, ds$. *If there exists a positive constant* c_0 *with* $c_0 c_p < 1$ *such that*

$$h(t) \leq c_0 (1+t)^{-p},$$

then, the function k *satisfies*

$$k(t) \leq \frac{c_0}{1 - c_0 c_p} (1+t)^{-p}.$$

Proof. See e.g [8]
Remark: The finiteness of the constant c_p can be found in [7, Lemma 7.4]. Due to this Lemma, in the remainder of this paper, we shall use (2..1)-(2..2) instead of (1..4)-(1..5). Let us denote by

$$(g \Box \varphi)(t) := \int_0^t g(t-s) |\varphi(t) - \varphi(s)|^2 ds.$$

The next lemma gives a important identity for the convolution product.

Lemma 2..2. *For* $g, \varphi \in C^1([0, \infty[: \mathbb{R})$ *we have*

$$(g * \varphi)\varphi_t \quad = \quad -\frac{1}{2} g(t) |\varphi(t)|^2 + \frac{1}{2} g' \Box \varphi - \frac{1}{2} \frac{d}{dt} \left[g \Box \varphi - \left(\int_0^t g(s) ds \right) |\varphi|^2 \right].$$

Proof. Differentiating the term $g \Box \varphi$ our conclusion follows. ∎

The first order energy of the coupled system (1..1)-(1..6) is given by

$$\begin{aligned} E(t, u, v) \quad := \quad & \frac{1}{2} \int_\Omega \Big(|u_t|^2 + \beta_1 |\nabla u|^2 \Big) dx + \frac{\tau_1 \beta_1}{2} \int_{\Gamma_1} \Big(k_1(t) |u|^2 - k_1' \Box u \Big) d\Gamma \\ & + \frac{1}{2} \int_\Omega \Big(|v_t|^2 + \beta_2 |\nabla v|^2 \Big) dx + \frac{\tau_2 \beta_2}{2} \int_{\Gamma_1} \Big(k_2(t) |v|^2 - k_2' \Box v \Big) d\Gamma \\ & + \int_\Omega F(u - v) dx. \end{aligned}$$

The well-posedness of the system (1..1)-(1..6) is given by the following theorem.

Theorem 2..1. *Let $k_i \in C^2(\mathbb{R}^+)$ such that*

$$k_i, -k_i', k_i'' \geq 0 \quad for \quad i = 1, 2.$$

If $(u_0, v_0) \in (H^2(\Omega) \cap V)^2$ and $(u_1, v_1) \in V^2$ satisfy the compatibility conditions

$$\frac{\partial u_0}{\partial \nu} + \tau_1 u_1 = 0 \quad on \quad \Gamma, \tag{2..3}$$

$$\frac{\partial v_0}{\partial \nu} + \tau_2 v_1 = 0 \quad on \quad \Gamma, \tag{2..4}$$

then there exists only one strong solution (u, v) for the coupled system (1..1)-(1..6) satisfying

$$u, v \in L^\infty(0, T; H^2(\Omega) \cap V) \cap W^{1,\infty}(0, T; V) \cap W^{2,\infty}(0, T; L^2(\Omega)).$$

Proof. To prove this theorem we shall use the Galerkin method. Let $\{(w_j, \phi_j)\}$ be a complete orthogonal system of $V \times V$ such that

$$\{(u_0, v_0), (u_1, v_1)\} \subset \operatorname{Span}\{(w_0, \phi_0), (w_1, \phi_1)\}.$$

Let us consider the finite approximations

$$(u^m(t), v^m(t)) := \sum_{j=0}^{m} h_{j,m}(t)(w_j, \phi_j),$$

which are solutions of the following ordinary differential equations

$$\begin{aligned} &\int_\Omega u_{tt}^m w_j dx + \beta_1 \int_\Omega \nabla u^m \cdot \nabla w_j dx + \int_\Omega f(u^m - v^m) w_j dx \\ &= -\tau_1 \beta_1 \int_{\Gamma_1} \{u_t^m + k_1(0) u^m - k_1(t) u^m(0) + k_1' * u^m\} w_j dx, \end{aligned} \tag{2..5}$$

$$\begin{aligned} &\int_\Omega v_{tt}^m \phi_j dx + \beta_2 \int_\Omega \nabla v^m \cdot \nabla \phi_j dx - \int_\Omega f(u^m - v^m) \phi_j dx \\ &= -\tau_2 \beta_2 \int_{\Gamma_1} \{v_t^m + k_2(0) v^m - k_2(t) v^m(0) + k_2' * v^m\} \phi_j dx, \end{aligned} \tag{2..6}$$

for $0 \leq j \leq m$, satisfying the initial data

$$(u^m(0), v^m(0)) = (u_0, v_0), \quad (u_t^m(0), v_t^m(0)) = (u_1, v_1).$$

Standard results about ordinary differential equations guarantee that there exists only one solution of this approximate system. Our first step is to show that the approximate solutions remain bounded for any $m \in \mathbb{N}$. Indeed, let us multiply equations (2..5) and (2..6) by $h_{j,m}'(t)$, summing up the product results in j and using Lemma 2..2 we conclude that

$$\frac{d}{dt} E(t, u^m, v^m) \leq c E(0, u^m, v^m).$$

Taking into account the definition of the initial data of (u^m, v^m) we conclude that

$$E(t, u^m, v^m) \leq c, \quad \forall t \in [0, T], \quad \forall m \in \mathbb{N}. \tag{2..7}$$

Next, we shall find an estimate for the second order energy. First, let us estimate the initial data $u_{tt}^m(0)$ and $v_{tt}^m(0)$ in the L^2-norm. Letting $t \to 0^+$ in the equations (2..5)-(2..6), multiplying the result by $h''_{j,m}(0)$ and using the compatibility conditions (2..3)-(2..4) we get

$$\int_\Omega |u_{tt}^m(0)|^2\, dx = \beta_1 \int_\Omega \Delta u_0 u_{tt}^m(0) dx - \int_\Omega f(u_0 - v_0) u_{tt}^m(0) dx,$$
$$\int_\Omega |v_{tt}^m(0)|^2\, dx = \beta_2 \int_\Omega \Delta v_0 v_{tt}^m(0) dx + \int_\Omega f(u_0 - v_0) v_{tt}^m(0) dx.$$

Since $(u_0, v_0) \in [H^2(\Omega)]^2$, the growth hypothesis for the function f together with the Sobolev's imbedding imply that $f(u_0 - v_0) \in L^2(\Omega)$. Hence

$$\int_\Omega \left(|u_{tt}^m(0)|^2 + |v_{tt}^m(0)|^2 \right) dx \leq M_1, \quad \forall m \in \mathbb{N}. \tag{2..8}$$

Differentiating equation (2..5) with respect to the time, multiplying by $h''_{j,m}(t)$ and summing up the product results in j, we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{ \int_\Omega \left(|u_{tt}^m|^2 + \beta_1 |\nabla u_t^m|^2 \right) dx + \tau_1\beta_1 \int_{\Gamma_1} \left(k_1(t)|u_t^m|^2 - k_1' \Box u_t^m \right) d\Gamma \right\} \\
&= - \int_\Omega f'(u^m - v^m)(u_t^m - v_t^m) u_{tt}^m dx - \tau_1\beta_1 \int_{\Gamma_1} |u_{tt}^m|^2 dx \\
&+ \frac{\tau_1\beta_1}{2} \int_{\Gamma_1} \left(k_1'(t)|u_t^m|^2 - k_1'' \Box u_t^m \right) d\Gamma.
\end{aligned}$$

Similarly we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{ \int_\Omega \left(|v_{tt}^m|^2 + \beta_2 |\nabla v_t^m|^2 \right) dx + \tau_2\beta_2 \int_{\Gamma_1} \left(k_2(t)|v_t^m|^2 - k_2' \Box v_t^m \right) d\Gamma \right\} \\
&= \int_\Omega f'(u^m - v^m)(u_t^m - v_t^m) v_{tt}^m dx - \tau_2\beta_2 \int_{\Gamma_1} |v_{tt}^m|^2 dx \\
&+ \frac{\tau_2\beta_2}{2} \int_{\Gamma_1} \left(k_2'(t)|v_t^m|^2 - k_2'' \Box v_t^m \right) d\Gamma.
\end{aligned}$$

Denoting by

$$\begin{aligned}
E_0(t, u, v) \quad = \quad & \frac{1}{2} \int_\Omega \left(|u_t|^2 + \beta_1 |\nabla u|^2 \right) dx + \frac{\tau_1\beta_1}{2} \int_{\Gamma_1} \left(k_1(t)|u|^2 - k_1' \Box u \right) d\Gamma \\
& + \frac{1}{2} \int_\Omega \left(|v_t|^2 + \beta_2 |\nabla v|^2 \right) dx + \frac{\tau_2\beta_2}{2} \int_{\Gamma_1} \left(k_2(t)|v|^2 - k_2' \Box v \right) d\Gamma,
\end{aligned}$$

we find that

$$\begin{aligned}
\frac{d}{dt} E_0(t, u_t^m, v_t^m) \quad = \quad & -\tau_1\beta_1 \int_{\Gamma_1} |u_{tt}^m|^2 dx + \frac{\tau_1\beta_1}{2} \int_{\Gamma_1} \left(k_1'(t)|u_t^m|^2 - k_1'' \Box u_t^m \right) d\Gamma \\
& -\tau_2\beta_2 \int_{\Gamma_1} |v_{tt}^m|^2 dx + \frac{\tau_2\beta_2}{2} \int_{\Gamma_1} \left(k_2'(t)|v_t^m|^2 - k_2'' \Box v_t^m \right) d\Gamma \\
& - \int_\Omega f'(u^m - v^m)(u_t^m - v_t^m) u_{tt}^m dx \\
& + \int_\Omega f'(u^m - v^m)(u_t^m - v_t^m) v_{tt}^m dx.
\end{aligned} \tag{2..9}$$

Let us take $p_n = \frac{2n}{n-2}$. From the growth condition of the functions f and the Sobolev imbedding we have

$$\begin{aligned}
&\int_\Omega f'(u^m - v^m)u_t^m u_{tt}^m dx \le c\int_\Omega (1 + 2|u^m - v^m|^{\rho-1})|u_t^m||u_{tt}^m|dx \\
&\le c\left[\int_\Omega (1 + 2|u^m - v^m|^{\rho-1})^n dx\right]^{\frac{1}{n}} \left[\int_\Omega |u_t^m|^{p_n} dx\right]^{\frac{1}{p_n}} \left[\int_\Omega |u_{tt}^m|^2 dx\right]^{\frac{1}{2}} \\
&\le c\left[\int_\Omega (1 + |\nabla u^m - \nabla v^m|^2) dx\right]^{\frac{\rho-1}{2}} \left[\int_\Omega |\nabla u_t^m|^2 dx\right]^{\frac{1}{2}} \left[\int_\Omega |u_{tt}^m|^2 dx\right]^{\frac{1}{2}}.
\end{aligned}$$

Taking into account the first estimate (2..7) we conclude that

$$\begin{aligned}
\int_\Omega f_1'(u^m)u_t^m u_{tt}^m dx \quad &\le \quad c\left[\int_\Omega |\nabla u_t^m|^2 dx\right]^{\frac{1}{2}} \left[\int_\Omega |u_{tt}^m|^2 dx\right]^{\frac{1}{2}} \\
&\le c\left\{\int_\Omega |\nabla u_t^m|^2 dx + \int_\Omega |u_{tt}^m|^2 dx\right\}. \qquad (2..10)
\end{aligned}$$

Similarly we get

$$-\int_\Omega f'(u^m - v^m)v_t^m u_{tt}^m dx \le c\left\{\int_\Omega |\nabla v_t^m|^2 dx + \int_\Omega |u_{tt}^m|^2 dx\right\}, \qquad (2..11)$$

$$\int_\Omega f'(u^m - v^m)u_t^m v_{tt}^m dx \le c\left\{\int_\Omega |\nabla u_t^m|^2 dx + \int_\Omega |v_{tt}^m|^2 dx\right\}, \qquad (2..12)$$

$$-\int_\Omega f'(u^m - v^m)v_t^m v_{tt}^m dx \le c\left\{\int_\Omega |\nabla v_t^m|^2 dx + \int_\Omega |v_{tt}^m|^2 dx\right\}. \qquad (2..13)$$

Substitution of the inequalities (2..10)-(2..13) into (2..9) we get

$$\frac{d}{dt}E_0(t, u_t^m, v_t^m) \quad \le \quad c_2\left\{\int_\Omega |u_{tt}^m|^2 + |\nabla u_t^m|^2 dx + \int_\Omega |v_{tt}^m|^2 + |\nabla v_t^m|^2 dx\right\}.$$

Integrating with respect to the time and applying Gronwall's inequality we conclude that

$$E_0(t, u_t^m, v_t^m) \le c, \quad \forall t \in [0, T], \quad \forall m \in \mathbb{N}. \qquad (2..14)$$

Now, from the estimates (2..7) and (2..14) and those of the Lions-Aubin's compactness Theorem we can pass to the limit in (2..5)-(2..6). The rest of the proof is a matter of routine. ■

3. Exponential Decay

In this section we shall study the asymptotic behavior of the solutions of system (1..1)-(1..6) when the resolvent kernels k_1 and k_2 are exponentially decreasing, that is, there exist positive constants b_1, b_2 such that

$$k_i(0) > 0, \quad k_i'(t) \le -b_1 k_i(t), \quad k_i''(t) \ge -b_2 k_i'(t) \quad \text{for} \quad i = 1, 2. \qquad (3..1)$$

Note that this conditions implies that

$$k_i(t) \le k_i(0)e^{-b_1 t} \quad \text{for} \quad i = 1, 2.$$

Our point of departure will be to establish some inequalities for the strong solution of the coupled system (1..1)- (1..6). Let us denote $E(t) = E(t, u, v)$.

Lemma 3..1. *Any strong solution (u,v) of the system (1..1)-(1..6) satisfies*

$$\begin{aligned}\frac{d}{dt}E(t) &\le -\frac{\tau_1\beta_1}{2}\int_{\Gamma_1}\Big(|u_t|^2+k_1''\Box u-k_1'(t)|u|^2+|k_1(t)u_0|^2\Big)\,d\Gamma\\ &\quad -\frac{\tau_2\beta_2}{2}\int_{\Gamma_1}\Big(|v_t|^2+k_2''\Box v-k_2'(t)|v|^2+|k_2(t)v_0|^2\Big)\,d\Gamma.\end{aligned}$$

Proof. Multiplying the equation (1..1) by u_t and integrating by parts over Ω we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega\Big(|u_t|^2+\beta_1|\nabla u|^2\Big)dx+\int_\Omega f(u-v)u_t dx=\beta_1\int_{\Gamma_1}\frac{\partial u}{\partial\nu}u_t d\Gamma.$$

Similarly we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega\Big(|v_t|^2+\beta_2|\nabla v|^2\Big)dx-\int_\Omega f(u-v)v_t dx=\beta_2\int_{\Gamma_1}\frac{\partial v}{\partial\nu}v_t d\Gamma.$$

Summing the above identities, substituting the boundary terms by (2..1)-(2..2) and using Lemma 2..2 our conclusion follows. ∎

Let us consider the following binary operator

$$(k\diamond h)(t):=\int_0^t k(t-s)\Big(h(t)-h(s)\Big)\,ds.$$

Then applying Hölder's inequality we have, for $0\le\mu\le 1$

$$|(k\diamond h)(t)|^2\le\left[\int_0^t |k(s)|^{2(1-\mu)}\,ds\right](|k|^{2\mu}\Box h)(t). \tag{3..2}$$

Let us introduce the following functionals

$$\begin{aligned}\mathcal{N}(t) &:= \int_\Omega\Big(|u_t|^2+\beta_1|\nabla u|^2+|v_t|^2+\beta_2|\nabla v|^2+F(u-v)\Big)dx,\\ \psi(t) &:= \int_\Omega\left\{m\cdot\nabla u+\left(\frac{n}{2}-\theta\right)u\right\}u_t dx+\int_\Omega\left\{m\cdot\nabla v+\left(\frac{n}{2}-\theta\right)v\right\}v_t dx,\end{aligned}$$

where θ is a small positive constant. The following lemma plays an important role for the construction of the Lyapunov functional desired.

Lemma 3..2. *For any strong solution of the system (1..1)-(1..6) we get*

$$\begin{aligned}\frac{d}{dt}\psi(t)\le -\frac{\theta}{2}\mathcal{N}(t)&+C\int_{\Gamma_1}\Big(|u_t|^2+|k_1(t)u|^2+|k_1'\diamond u|^2+|k_1(t)u_0|^2\Big)\,d\Gamma\\ &+C\int_{\Gamma_1}\Big(|v_t|^2+|k_2(t)v|^2+|k_2'\diamond v|^2+|k_2(t)v_0|^2\Big)\,d\Gamma,\end{aligned}$$

for some positive constant C.

Proof. From equation (1..1) we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u_t\left\{m\cdot\nabla u+\left(\frac{n}{2}-\theta\right)u\right\}dx\\
&\quad= \int_\Omega u_t m\cdot\nabla u_t dx+\left(\frac{n}{2}-\theta\right)\int_\Omega |u_t|^2dx\\
&\qquad+\beta_1\int_\Omega \Delta u m\cdot\nabla u dx+\beta_1\left(\frac{n}{2}-\theta\right)\int_\Omega \Delta u u dx\\
&\qquad-\left(\frac{n}{2}-\theta\right)\int_\Omega f(u-v)udx-\int_\Omega f(u-v)m\cdot\nabla udx.
\end{aligned}$$

Performing an integration by parts we get

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u_t\left\{m\cdot\nabla u+\left(\frac{n}{2}-\theta\right)u\right\}dx\\
&\quad\le \frac{1}{2}\int_{\Gamma_1} m\cdot\nu|u_t|^2d\Gamma-\theta\int_\Omega |u_t|^2dx\\
&\qquad+\beta_1\int_{\Gamma_1}\frac{\partial u}{\partial\nu}\left\{m\cdot\nabla u+\left(\frac{n}{2}-\theta\right)u\right\}d\Gamma\\
&\qquad-\frac{\beta_1}{2}\int_{\Gamma_1} m\cdot\nu|\nabla u|^2d\Gamma-\beta_1(1-\theta)\int_\Omega|\nabla u|^2dx\\
&\qquad-\left(\frac{n}{2}-\theta\right)\int_\Omega f(u-v)udx-\int_\Omega f(u-v)m\cdot\nabla udx.
\end{aligned}$$

Similarly, using equation (1..2) instead of (1..1) we get

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega v_t\left\{m\cdot\nabla v+\left(\frac{n}{2}-\theta\right)v\right\}dx\\
&\quad\le \frac{1}{2}\int_{\Gamma_1} m\cdot\nu|v_t|^2d\Gamma-\theta\int_\Omega |v_t|^2dx\\
&\qquad+\beta_2\int_{\Gamma_1}\frac{\partial v}{\partial\nu}\left\{m\cdot\nabla v+\left(\frac{n}{2}-\theta\right)v\right\}d\Gamma\\
&\qquad-\frac{\beta_2}{2}\int_{\Gamma_1} m\cdot\nu|\nabla v|^2d\Gamma-\beta_2(1-\theta)\int_\Omega|\nabla v|^2dx\\
&\qquad+\left(\frac{n}{2}-\theta\right)\int_\Omega f(u-v)vdx+\int_\Omega f(u-v)m\cdot\nabla vdx.
\end{aligned}$$

Adding up these two last inequalities and taking into account that f is superlinear we arrive at

$$\begin{aligned}
\frac{d}{dt}\psi(t)\le{}& \frac{1}{2}\int_{\Gamma_1} m\cdot\nu(|u_t|^2+|v_t|^2)d\Gamma-\theta\int_\Omega\Big(|u_t|^2+|v_t|^2\Big)dx\\
&-(1-\theta)\int_\Omega\beta_1|\nabla u|^2dx-(1-\theta)\int_\Omega\beta_2|\nabla v|^2dx\\
&+\beta_1\int_{\Gamma_1}\frac{\partial u}{\partial\nu}\left\{m\cdot\nabla u+\left(\frac{n}{2}-\theta\right)u\right\}d\Gamma\\
&+\beta_2\int_{\Gamma_1}\frac{\partial v}{\partial\nu}\left\{m\cdot\nabla v+\left(\frac{n}{2}-\theta\right)v\right\}d\Gamma\\
&-\frac{1}{2}\int_{\Gamma_1} m\cdot\nu\Big(\beta_1|\nabla u|^2+\beta_2|\nabla v|^2\Big)d\Gamma\\
&-(\frac{n}{2}-\theta)(2+\delta)\int_\Omega F(u-v)dx+n\int_\Omega F(u-v)dx.
\end{aligned}$$

Taking θ small enough we obtain

$$\begin{aligned}
\frac{d}{dt}\psi(t) \leq & -\theta\mathcal{N}(t) + \frac{1}{2}\int_{\Gamma_1} m\cdot\nu(|u_t|^2 + |v_t|^2)d\Gamma \\
& - \frac{1}{2}\int_{\Gamma_1} m\cdot\nu\Big(\beta_1|\nabla u|^2 + \beta_2|\nabla v|^2\Big)d\Gamma \\
& + \beta_1\int_{\Gamma_1}\frac{\partial u}{\partial\nu}\left\{m\cdot\nabla u + \left(\frac{n}{2}-\theta\right)u\right\}d\Gamma \\
& + \beta_2\int_{\Gamma_1}\frac{\partial v}{\partial\nu}\left\{m\cdot\nabla v + \left(\frac{n}{2}-\theta\right)v\right\}d\Gamma. \qquad (3..3)
\end{aligned}$$

Now, we analyze some boundary term of the above inequality. Applying Young's and Poincaré's inequalities we have, for $\epsilon > 0$

$$\begin{aligned}
& \beta_1\int_{\Gamma_1}\frac{\partial u}{\partial\nu}\left\{m\cdot\nabla u + \left(\frac{n}{2}-\theta\right)u\right\}d\Gamma \\
& \qquad \leq \epsilon\int_{\Gamma_1}\left\{|m\cdot\nabla u|^2 + \left(\frac{n}{2}-\theta\right)^2|u|^2\right\}d\Gamma + C_\epsilon\int_{\Gamma_1}\left|\frac{\partial u}{\partial\nu}\right|^2 d\Gamma \\
& \qquad \leq \epsilon C\left\{\int_{\Gamma_1} m\cdot\nu|\nabla u|^2 d\Gamma + \mathcal{N}(t)\right\} + C_\epsilon\int_{\Gamma_1}\left|\frac{\partial u}{\partial\nu}\right|^2 d\Gamma.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
& \beta_2\int_{\Gamma_1}\frac{\partial v}{\partial\nu}\left\{m\cdot\nabla v + \left(\frac{n}{2}-\theta\right)v\right\}d\Gamma \\
& \qquad \leq \epsilon C\left\{\int_{\Gamma_1} m\cdot\nu|\nabla v|^2 d\Gamma + \mathcal{N}(t)\right\} + C_\epsilon\int_{\Gamma_1}\left|\frac{\partial v}{\partial\nu}\right|^2 d\Gamma.
\end{aligned}$$

By substitution of these last inequalities into (3..3) with ϵ small and taking into account that the boundary conditions (2..1)-(2..2) can be written as

$$\begin{aligned}
\frac{\partial u}{\partial\nu} &= -\tau_1\{u_t + k_1(t)u - k_1'\diamond u - k_1(t)u_0\}, \\
\frac{\partial v}{\partial\nu} &= -\tau_2\{v_t + k_2(t)v - k_2'\diamond v - k_2(t)v_0\},
\end{aligned}$$

our conclusion follows. ■

To show that the energy decays exponentially we shall need the following lemma.

Lemma 3..3. *Let f be a real positive function of class C^1. If there exist positive constants γ_0, γ_1 and c_0 such that*

$$f'(t) \leq -\gamma_0 f(t) + c_0 e^{-\gamma_1 t},$$

then there exist positive constants γ and c such that

$$f(t) \leq (f(0) + c)e^{-\gamma t}.$$

Proof. See e. g. [10].
Finally, we shall show the main result of this section.

Theorem 3..1. *Let us take $(u_0, v_0) \in V^2$ and $(u_1, v_1) \in [L^2(\Omega)]^2$. If the resolvent kernels k_1, k_2 satisfy (3..1), then there exist positive constants C and γ such that*

$$E(t) \leq CE(0)e^{-\gamma t},$$

for all $t \geq 0$.

Proof. We shall prove this result for strong solutions, that is, for solutions with initial data $(u_0, v_0) \in (H^2(\Omega) \cap V)^2$ and $(u_1, v_1) \in V^2$ satisfying the compatibility conditions (2..3)- (2..4). Our conclusion follows from standard density arguments. Using hypothesis (3..1) in Lemma 3..1 we get

$$\begin{aligned} \frac{d}{dt}E(t) \quad &\leq \quad -\frac{\tau_1\beta_1}{2}\int_{\Gamma_1}\left(|u_t|^2 - b_2 k_1'\Box u + b_1 k_1(t)|u|^2 + |k_1(t)u_0|^2\right) d\Gamma \\ &\quad -\frac{\tau_2\beta_2}{2}\int_{\Gamma_1}\left(|v_t|^2 - b_2 k_2'\Box v + b_1 k_2(t)|v|^2 + |k_2(t)v_0|^2\right) d\Gamma. \end{aligned}$$

On the other hand applying inequality (3..2) with $\mu = 1/2$ in Lemma 3..2 we obtain

$$\begin{aligned} \frac{d}{dt}\psi(t) \leq -\frac{\theta}{2}\mathcal{N}(t) + C\int_{\Gamma_1}\left(|u_t|^2 + k_1(t)|u|^2 - k_1'\Box u + |k_1(t)u_0|^2\right) d\Gamma \\ + C\int_{\Gamma_1}\left(|v_t|^2 + k_2(t)|v|^2 - k_2'\Box v + |k_2(t)v_0|^2\right) d\Gamma. \end{aligned}$$

Let us introduce the Lyapunov functional

$$\mathcal{L}(t) := NE(t) + \psi(t), \tag{3..4}$$

with $N > 0$. Taking N large, the previous inequalities imply that

$$\frac{d}{dt}\mathcal{L}(t) \leq -\frac{\theta}{2}E(t) + 2NR^2(t)E(0),$$

where $R(t) = k_1(t) + k_2(t)$. Moreover, using Young's inequality and taking N large we find that

$$\frac{N}{2}E(t) \leq \mathcal{L}(t) \leq 2NE(t). \tag{3..5}$$

From this inequality we conclude that

$$\frac{d}{dt}\mathcal{L}(t) \leq -\frac{\theta}{2}\mathcal{L}(t) + 2NR^2(t)E(0),$$

from which follows, in view of Lemma 3..3 and the exponential decay of k_1, k_2, that

$$\mathcal{L}(t) \leq \{\mathcal{L}(0) + c\}e^{-\gamma_1 t},$$

for some positive constants c, γ. From the inequality (3..5) our conclusion follows. ∎

4. Polynomial decay

Here our attention will be focused on the uniform rate of decay when the resolvent kernels k_1 and k_2 decays polynomially like $(1+t)^{-p}$. In this case we will show that the solution also decays polynomially with the same rate. Therefore, we will assume that the resolvent kernels $k_1,\ k_2$ satisfy

$$k_i(0)>0, \quad k_i'(t)\leq -b_1 k_i(t)^{1+\frac{1}{p}}, \quad k_i''(t)\geq b_2[-k_i'(t)]^{1+\frac{1}{p+1}} \quad \text{for} \quad i=1,2, \quad (4..1)$$

for some $p>1$ and some positive constants $b_1,\ b_2$. The following lemma will play an important role in the sequel.

Lemma 4..1. *Let (u,v) be a solution of system (1..1)-(1..6) and let us denote $(\phi_1,\phi_2)=(u,v)$. Then, for $p>1$, $0<r<1$ and $t\geq 0$, we have*

$$\left(\int_{\Gamma_1}|k_i'|\Box\phi_i d\Gamma\right)^{\frac{1+(1-r)(p+1)}{(1-r)(p+1)}}\leq 2\left(\int_0^t |k_i'(s)|^r ds\|\phi_i\|^2_{L^\infty(0,t;L^2(\Gamma_1))}\right)^{\frac{1}{(1-r)(p+1)}}\int_{\Gamma_1}|k_i'|^{1+\frac{1}{p+1}}\Box\phi_i d\Gamma,$$

while for $r=0$ we get

$$\left(\int_{\Gamma 1}|k_i'|\Box\phi_i d\Gamma\right)^{\frac{p+2}{p+1}}\leq 2\left(\int_0^t \|\phi_i(s,.)\|^2_{L^2(\Gamma_1)}ds+t\|\phi_i(s,.)\|^2_{L^2(\Gamma_1)}\right)^{\frac{1}{p+1}}\int_{\Gamma_1}|k_i'|^{1+\frac{1}{p+1}}\Box\phi_i d\Gamma,$$

for $i=1,2$.

Proof. See e. g. [9].

Lemma 4..2. *Let $f\geq 0$ be a differentiable function satisfying*

$$f'(t)\leq -\frac{c_1}{f(0)^{\frac{1}{\alpha}}}f(t)^{1+\frac{1}{\alpha}}+\frac{c_2}{(1+t)^\beta}f(0) \quad \textit{for} \quad t\geq 0,$$

for some positive constants c_1,c_2, α and β such that

$$\beta\geq\alpha+1.$$

Then there exists a constant $c>0$ such that

$$f(t)\leq\frac{c}{(1+t)^\alpha}f(0) \quad \textit{for} \quad t\geq 0.$$

Proof. See e. g. [10].

The polynomial decay of the energy is given by the following theorem.

Theorem 4..1. *Let us take $(u_0,v_0)\in V^2$ and $(u_1,v_1)\in[L^2(\Omega)]^2$. If the resolvent kernels $k_1,\ k_2$ satisfy the conditions (4..1), then there exists a positive constant c such that*

$$E(t)\leq\frac{c}{(1+t)^{p+1}}E(0).$$

Proof. We shall prove this result for strong solutions, that is, for solutions with initial data $(u_0, v_0) \in (H^2(\Omega) \cap V)^2$ and $(u_1, v_1) \in V^2$ satisfying the compatibility conditions (2..3)- (2..4). Our conclusion will follow by standard density arguments. We use some estimates of the previous section which are independent of the behavior of the resolvent kernels $k_1,\ k_2$. Using hypothesis (4..1) in Lemma 3..1 yields

$$\begin{aligned}\frac{d}{dt}E(t) \quad &\le \quad -\frac{\tau_1\beta_1}{2}\int_{\Gamma_1}\left(|u_t|^2 + b_2[-k_1']^{1+\frac{1}{p+1}}\Box u + b_1 k_1^{1+\frac{1}{p}}(t)|u|^2 + |k_1(t)u_0|^2\right)d\Gamma\\ &\quad -\frac{\tau_2\beta_2}{2}\int_{\Gamma_1}\left(|v_t|^2 + b_2[-k_2']^{1+\frac{1}{p+1}}\Box v + b_1 k_2^{1+\frac{1}{p}}(t)|v|^2 + |k_2(t)v_0|^2\right)d\Gamma.\end{aligned}$$

Applying inequality (3..2) with $\mu = \frac{p+2}{2(p+1)}$ and using hypothesis (4..1) we obtain the following estimates

$$|k_1' \diamond u|^2 \le C[-k_1']^{1+\frac{1}{p+1}}\Box u, \quad |k_2' \diamond v|^2 \le C[-k_2']^{1+\frac{1}{p+1}}\Box v.$$

Using the above inequalities in Lemma 3..2 yields

$$\begin{aligned}\frac{d}{dt}\psi(t) \le -\frac{\theta}{2}\mathcal{N}(t) &+ C\int_{\Gamma_1}\left(|u_t|^2 + k_1^{1+\frac{1}{p}}(t)|u|^2 + [-k_1']^{1+\frac{1}{p+1}}\Box u + |k_1(t)u_0|^2\right)d\Gamma\\ &+ C\int_{\Gamma_1}\left(|v_t|^2 + k_2^{1+\frac{1}{p}}(t)|v|^2 + [-k_2']^{1+\frac{1}{p+1}}\Box v + |k_2(t)v_0|^2\right)d\Gamma.\end{aligned}$$

In these conditions, taking N large, the Lyapunov functional defined in (3..4) satisfies

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) \quad &\le \quad -\frac{\theta}{2}\mathcal{N}(t) + 2NR^2(t)E(0)\\ &\quad -\frac{Nc_2}{2}\left\{\int_{\Gamma_1}[-k_1']^{1+\frac{1}{p+1}}\Box u d\Gamma + \int_{\Gamma_1}[-k_2']^{1+\frac{1}{p+1}}\Box v d\Gamma\right\}. \qquad (4..2)\end{aligned}$$

Let us fix $0 < r < 1$ such that $\frac{1}{p+1} < r < \frac{p}{p+1}$. From (4..1) we have that

$$\int_0^\infty |k_i'|^r \le c\int_0^\infty \frac{1}{(1+t)^{r(p+1)}} < \infty \quad \text{for} \quad i = 1, 2.$$

Using this estimate in Lemma 4..1 we get

$$\int_{\Gamma_1}[-k_1']^{1+\frac{1}{p+1}}\Box u d\Gamma \ge cE(0)^{-\frac{1}{(1-r)(p+1)}}\left(\int_{\Gamma_1}[-k_1']\Box u d\Gamma\right)^{1+\frac{1}{(1-r)(p+1)}}, \qquad (4..3)$$

$$\int_{\Gamma_1}[-k_2']^{1+\frac{1}{p+1}}\Box v\ d\Gamma \ge cE(0)^{-\frac{1}{(1-r)(p+1)}}\left(\int_{\Gamma_1}[-k_2']\Box v d\Gamma\right)^{1+\frac{1}{(1-r)(p+1)}}. \qquad (4..4)$$

On the other hand, from the Trace Theorem we have

$$E(t)^{1+\frac{1}{(1-r)(p+1)}} \le cE(0)^{\frac{1}{(1-r)(p+1)}}\mathcal{N}(t). \qquad (4..5)$$

Substitution of (4..3)-(4..5) into (4..2) we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) &\le -cE(0)^{-\frac{1}{(1-r)(p+1)}}E(t)^{1+\frac{1}{(1-r)(p+1)}} + 2NR^2(t)E(0)\\ &- cE(0)^{-\frac{1}{(1-r)(p+1)}}\left\{\left(\int_{\Gamma_1}[-k_1']\Box u d\Gamma\right)^{1+\frac{1}{(1-r)(p+1)}} + \left(\int_{\Gamma_1}[-k_2']\Box v d\Gamma\right)^{1+\frac{1}{(1-r)(p+1)}}\right\}.\end{aligned}$$

Taking into account the inequality (3..5) we conclude that

$$\frac{d}{dt}\mathcal{L}(t) \leq -\frac{c}{\mathcal{L}(0)^{\frac{1}{(1-r)(p+1)}}}\mathcal{L}(t)^{1+\frac{1}{(1-r)(p+1)}} + 2NR^2(t)E(0),$$

for some $c > 0$, from which follows, applying Lemma 4..2, that

$$\mathcal{L}(t) \leq \frac{c}{(1+t)^{(1-r)(p+1)}}\mathcal{L}(0).$$

Since $(1-r)(p+1) > 1$ we get, for $t \geq 0$, the following bounds

$$\begin{aligned} t\|u\|^2_{L^2(\Gamma_1)} + t\|v\|^2_{L^2(\Gamma_1)} \leq t\mathcal{L}(t) \quad &< \quad \infty, \\ \int_0^t \left(\|u\|^2_{L^2(\Gamma_1)} + \|v\|^2_{L^2(\Gamma_1)}\right)\, ds \leq c\int_0^t \mathcal{L}(t)\, ds \quad &< \quad \infty. \end{aligned}$$

Using the above estimates in Lemma 4..1 with $r = 0$ we get

$$\begin{aligned} \int_{\Gamma_1} [-k_1']^{1+\frac{1}{p+1}}\Box u d\Gamma \geq \frac{c}{E(0)^{\frac{1}{p+1}}}\left(\int_{\Gamma_1}[-k_1']\Box u d\Gamma\right)^{1+\frac{1}{p+1}}, \\ \int_{\Gamma_1} [-k_2']^{1+\frac{1}{p+1}}\Box v d\Gamma \geq \frac{c}{E(0)^{\frac{1}{p+1}}}\left(\int_{\Gamma_1}[-k_2']\Box v d\Gamma\right)^{1+\frac{1}{p+1}}. \end{aligned}$$

Using these inequalities instead of (4..3)-(4..4) and reasoning in the same way as above we conclude that

$$\frac{d}{dt}\mathcal{L}(t) \leq -\frac{c}{\mathcal{L}(0)^{\frac{1}{p+1}}}\mathcal{L}(t)^{1+\frac{1}{p+1}} + 2NR^2(t)E(0).$$

Applying Lemma 4..2 again, we obtain

$$\mathcal{L}(t) \leq \frac{c}{(1+t)^{p+1}}\mathcal{L}(0).$$

Finally, from (3..5) we conclude

$$E(t) \leq \frac{c}{(1+t)^{p+1}}E(0),$$

which completes the proof. ■

5. Final Comments

The methods explored in this paper can be used to solve the following nonlinear coupled systems:

$$\left\{ \begin{array}{c} u_{tt} - \Delta u + \int_0^t g_1(t-s)\Delta u(s)ds + f(u-v) = 0 \text{ in } \Omega \times (0,\infty), \\ v_{tt} - \Delta v + \int_0^t g_2(t-s)\Delta v(s)ds - f(u-v) = 0 \text{ in } \Omega \times (0,\infty), \\ u = v = 0 \text{ on } \Gamma \times (0,\infty), \\ (u(0,x), v(0,x)) = (u_0(x), v_0(x)), \quad (u_t(0,x), v_t(0,x)) = (u_1(x), v_1(x)) \text{ in } \Omega. \end{array} \right. \tag{5..1}$$

$$
\left\{
\begin{array}{c}
u_{tt} + \Delta^2 u - M\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)\Delta u - \Delta u_t + f(u-v) = 0 \\
\text{in } \Omega \times (0,\infty) \\
v_{tt} + \Delta^2 v - M\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)\Delta v - \Delta v_t - f(u-v) = 0 \\
\text{in } \Omega \times (0,\infty) \\
u = v = \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \\
\text{on } \Gamma_0 \times (0,\infty) \\
\Delta u + \int_0^t g_1(t-s)((M\left(\|\nabla u(s)\|_2^2 + \|\nabla v(s)\|_2^2\right)\frac{\partial u}{\partial \nu}(s) + \frac{\partial u_t}{\partial \nu}(s))ds = 0 \\
\text{on } \Gamma_1 \times (0,\infty) \\
\Delta v + \int_0^t g_2(t-s)((M\left(\|\nabla u(s)\|_2^2 + \|\nabla v(s)\|_2^2\right)\frac{\partial v}{\partial \nu}(s) + \frac{\partial v_t}{\partial \nu}(s))ds = 0 \\
\text{on } \Gamma_1 \times (0,\infty) \\
(u(0,x), v(0,x)) = (u_0(x), v_0(x)) \text{ in } \Omega \\
(u_t(0,x), v_t(0,x)) = (u_1(x), v_1(x)) \text{ in } \Omega
\end{array}
\right.
\tag{5..2}
$$

$$
\left\{
\begin{array}{c}
K_1(x,t)u_{tt} - M\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)\Delta u + K_3(x,t)u_t - \Delta u_t + f(u-v) = 0, \\
\text{in } \Omega \times (0,\infty), \\
K_2(x,t)v_{tt} - M\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)\Delta v + K_4(x,t)v_t - \Delta v_t - f(u-v) = 0, \\
\text{in } \Omega \times (0,\infty), \\
u = v = 0 \text{ on } \Gamma_0 \times (0,\infty), \\
u + \int_0^t g_1(t-s)((M\left(\|\nabla u(s)\|_2^2 + \|\nabla v(s)\|_2^2\right)\frac{\partial u}{\partial \nu}(s) + \frac{\partial u_t}{\partial \nu}(s))ds = 0, \\
\text{on } \Gamma_1 \times (0,\infty), \\
v + \int_0^t g_2(t-s)((M\left(\|\nabla u(s)\|_2^2 + \|\nabla v(s)\|_2^2\right)\frac{\partial v}{\partial \nu}(s) + \frac{\partial v_t}{\partial \nu}(s))ds = 0, \\
\text{on } \Gamma_1 \times (0,\infty), \\
(u(0,x), v(0,x)) = (u_0(x), v_0(x)) \text{ in } \Omega, \\
(u_t(0,x), v_t(0,x)) = (u_1(x), v_1(x)) \text{ in } \Omega
\end{array}
\right.
\tag{5..3}
$$

where $K_i(x,t) \geq 0$ and $K_j(x,0) \geq d > 0$ a.e in $\Omega \times (0,\infty)$ for all $i = 1,2$ and $j = 3,4$ respectively. With regard to the problems (5..1), (5..2) and (5..3,) it is important to observe, however, that, as far as we are aware of, nonlinear memory terms acting in the boundary have never been considered in the literature. Problem (5.3) concerns a system of equations, degenerate nonlinear and nonlinear boundary feedback combined with a nonlinear memory source term which requires new arguments to overcome the difficulties. The authors of the paper already obtained the results of global existence, uniqueness and the exponential and polynomial declines respectively for the problems (5.1), (5.2) and (5..3). Three papers are in their final phase and will shortly be submitted for publication.

Now we would like to mention that the problem (1..1)-(1..6) as well as the corresponding problem for the systems (5..1), (5..2) and (5..3) in a bounded domain with moving boundary are interesting open problems.

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References

[1] M. Aassila, A note on the boundary stabilization of a compactly coupled system of wave equations, *Applied Mathematics Letters 12, 37-42 (1999).*

[2] M. Aassila, Strong Asymptotic Stability of a Compactly coupled System of Wave Equations. *Applied Mathematics Letters 14, 285-290 (2001).*

[3] F. Alabau, Stabilisation frontière indirecte de systèmes faiblement couplés. *C. R. Acad. Sci. Paris, 328 , 1015-1020 (1999).*

[4] V. Komornik and B. Rao, Boundary stabilization of compactly coupled wave equations. *Asymptotic Analysis 14, 339-359 (1997).*

[5] J. L. Lions, Quelques Méthodes de resolution de problèmes aux limites non linéaires. *Dunod Gauthiers Villars, Paris (1969).*

[6] H. P. Oquendo and M. L. Santos, Stability for a coupled system with memory condition at the boundary (submitted to *Asymptotic Analysis*).

[7] R. Racke, Lectures on nonlinear evolution equations. Initial value problems, *Aspect of Mathematics E19. Friedr. Vieweg* & *Sohn, Braunschweig/Wiesbaden (1992).*

[8] M. L. Santos, Decay rates for solutions of a Timoshenko system with a memory condition at the boundary, *Abstract and Applied Analysis, 7:10, 531-546 (2002).*

[9] M. L. Santos, Asymptotic behavior of solutions to wave equations with a memory condition at the boundary, *E. J. Diff. Eqs. 73, 1-11 (2001).*

[10] M. L. Santos, Decay rates for solutions of a system of wave equations with memory, *E. J. Diff. Eqs. 38, 1-17 (2002)* .

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