On some properties of the Hilbert transform in Euclidean space

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Dedicated to Fred Brackx and Frank Sommen

Abstract

First some properties are recalled of the Hilbert transform introduced in the 1990's within the framework of Clifford analysis. Furthermore algebraic and geometric characterizations are given for this operator to be unitary. Special attention is paid to the Hilbert transform on the unit sphere S^m and the hyperplane $\mathbb{R}^m$ in $\mathbb{R}^{m+1}$ and classical results in the plane are revisited.

1 Introduction

Although named after David Hilbert, The Hilbert transform on the real line and its properties were developed mainly by Titchmarsch and Hardy. The Hilbert transform appears naturally when studying the boundary behaviour of the Cauchy transform $\mathcal{C}f$ of a function $f \in L_2(\sum)$, $\sum$ being either the boundary of a bounded Lipschitz domain in $\mathbb{C}$ or the graph of a Lipschitz continuous function in $\mathbb{R}$. The most elaborated examples are the cases of the unit circle and the real line, leading to deep results in harmonic analysis (see e.g. [12]). The central formula establishing the relation between the boundary value of the Cauchy transform and the Hilbert transform is the so-called Plemelj-Sokhotzki formula.

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A higher dimensional analogue of the Hilbert transform appeared in the study of H^p-spaces of harmonic functions of several variables, more precisely when the relationship was established between the boundary values of solutions of the Riesz system in the upper half space $\mathbb{R}^{m+1}_+$ of $\mathbb{R}^{m+1}$ and the Riesz transforms R_j on $\mathbb{R}^m$, $j = 1, ..., m$ (see [15]).

In the mid-1980's, it became clear that Clifford analysis provided a natural framework for generalizing a lot of results from harmonic analysis in the plane to the higher dimensional case. The main tool used was the Cauchy transform which, by taking boundary values, led to a Plemelj-Sokhotzki type formula. In such a way, properties of the singular integral operator appearing in this formula could be studied by means of function theoretic methods (see e.g. [10] and [13]).

In 1978, Kerzman and Stein (see [11]) proved a fundamental property of the Cauchy transform $\mathcal{C}f$ of $f \in L_2(\sum)$, $\sum$ being the boundary of a bounded open domain Ω in $\mathbb{C}$ with C_∞-boundary. They discovered that the operator $\mathbb{A} = \mathcal{C} - \mathcal{C}^*$ is a compact infinitely smoothing operator on $L_2(\sum)$ and that the Hardy projection $\mathcal{C}$ and the Szegö-projection $\mathbb{P}$ of $L_2(\sum)$ onto the Hardy space $H^2(\sum)$ are related by the formula $\mathbb{P}(\mathbf{1} + \mathbb{A}) = \mathcal{C}$. Moreover, they showed that the disc is the only plane region for which the Szegö and Cauchy kernels coincide. Implicitly, this result also tells us that the Hilbert transform H on $L_2(\sum)$ is unitary if and only if Ω is a disc.

In the mid-1990's, the Kerzman-Stein formula $\mathbb{P}(\mathbf{1}+\mathbb{A}) = \mathcal{C}$ has been generalized to domains Ω in R^{m+1} by Calderbank (see [6]) and Cnops (see [7]). Their results are even valid in a much more general context, namely within the study of boundary value problems for Dirac operators on manifolds (see also [3]).

In the underlying paper we give some characterizations for the unitariness of the Hilbert transform (§2). In the following sections (§§3-4) we pay special attention to the cases of the unit sphere $\mathbb{S}^m$ and the hyperplane $\mathbb{R}^m$ in $\mathbb{R}^{m+1}$. Finally, in section 5 we show how classical results in the plane are included in our approach.
We dedicate this paper to Fred Brackx and Frank Sommen and this on the occasion of our retirement: they were our first two Ph.D-students in Clifford analysis. During the thirty five years, respectively the twenty five years, of intensive collaboration, strong ties of friendship were forged.

2 The Hilbert transform in Euclidean space

Throughout this paper we suppose that Ω is either an open bounded subset of $\mathbb{R}^{m+1}$ with C_∞-boundary $\sum$, or $\Omega = \mathbb{R}^{m+1}_+ = \{x = (x_0, x_1, ..., x_m) : x_m > 0\}$, the upper half space in $\mathbb{R}^{m+1}$ with boundary $\sum = \mathbb{R}^m$. The outward pointing unit normal at $y \in \sum$ will be denoted by $\nu(y)$.

The space $\mathbb{R}^{m+1}$ will be identified with the space of vectors $\mathbb{R}^{0,m+1}$ in the universal Clifford algebra constructed over the real vector space $\mathbb{R}^{m+1}$ equipped with a quadratic form of signature $(0, m+1)$, and this in the following way.

Let $e = (e_0, e_1, ..., e_m)$ be an orthonormal basis of $\mathbb{R}^{0,m+1}$. Then the non-commutative multiplication in $\mathbb{R}_{0,m+1}$ is governed by the rules

$$e_i^2 = -1,\ i = 0, 1, ..., m$$

and

$$e_i e_j + e_j e_i = 0,\ i \neq j.$$

A basis of $\mathbb{R}_{0,m+1}$ is given by the set $(e_A :\ A \subset \{0, 1, ..., m\})$, where for $A = \{i_1, i_2, ..., i_h\}$, $0 \leq i_1 < i_2 < ... < i_h \leq m$, $e_A = e_{i_1} e_{i_2} ... e_{i_h}$. Putting $e_\emptyset = 1$, the identity element of $\mathbb{R}_{0,m+1}$, we may embed $\mathbb{R}$ and $\mathbb{R}^{m+1}$ in $\mathbb{R}_{0,m+1}$ by identifying $\mathbb{R}$ with $\mathbb{R}1$ and $\mathbb{R}^{m+1}$ with $\mathrm{span}_{\mathbb{R}}(e_i : i = 0, 1, ..., m)$, i.e. $a \in \mathbb{R}$ and $x = (x_0, x_1, ..., x_m) \in \mathbb{R}^{m+1}$ correspond to $a1$ and $x = \sum_{i=0}^m x_i e_i$.

The conjugation $a \longrightarrow \bar{a}$ in $\mathbb{R}_{0,m+1}$ is the anti-involution defined by $\overline{e_i} = -e_i$, $i = 0, 1, ..., m$, and $\overline{ab} = \bar{b}\,\bar{a}$ for $a, b \in \mathbb{R}_{0,m+1}$.

In the complex Clifford algebra $\mathbb{C}_{m+1} = \mathbb{R}_{0,m+1} \otimes_{\mathbb{R}} \mathbb{C}$, the conjugation $a \longrightarrow \bar{a}$ may be introduced by taking the tensor product of the conjugation in $\mathbb{R}_{0,m+1}$ and the complex conjugation in $\mathbb{C}$.

A basis in $\mathbb{C}_{m+1}$ is given by the elements $e_A \otimes 1$, denoted by e_A for the sake of simplicity.

Denoting by $\mathcal{A}$ the algebra $\mathbb{R}_{0,m+1}$ or $\mathbb{C}_{m+1}$, an element $a \in \mathcal{A}$ may thus be written as

$$a = \sum_{k=0}^{m+1} [a]_k$$

where for $k \in \{0, 1, ..., m+1\}$, $[\ ]_k$ stands for the projection of $\mathcal{A}$ onto the subspace $\mathcal{A}^{(k)}$ of k-vectors, with $\mathcal{A}^{(k)} = \mathrm{span}\ (e_A : |A| = k)$.

Notice that for $x, y \in \mathbb{R}^{m+1}$,

$$xy = x \cdot y + x \wedge y$$

with

$$x \cdot y = -\sum_{i=0}^m x_i y_i$$

and

$$x \wedge y = \sum_{i<y} e_i e_j (x_i y_j - x_j y_i),$$

i.e. $xy \in \mathcal{A}^{(0)} \oplus \mathcal{A}^{(2)}$.

In particular

$$x^2 = -|x|^2.$$

Define the Dirac operator ∂_x in $\mathbb{R}^{m+1}$ by

$$\partial_x = \sum_{i=0}^m e_i \partial_{x_i}.$$

Then $\partial_x^2 = -\triangle_x, \triangle_x$ being the Laplacian in $\mathbb{R}^{m+1}$.
The operator ∂_x is strongly elliptic and its fundamental solution is given by

$$E(x) = \frac{1}{A_{m+1}} \frac{\bar{x}}{|x|^{m+1}},$$

where A_{m+1} is the area of the unit sphere S^m in $\mathbb{R}^{m+1}$.

In what follows, functions f will be considered which are defined in some subset $G \subset \mathbb{R}^{m+1}$ and which are $\mathcal{A}$-valued. We say that $f = \sum_A f_A e_A$ belongs to $C_1(G), L_2(G)$, etc..., if all of its components f_A belong the classical function spaces $C_1(G), L_2(G)$, etc... of $\mathbb{R}$-or $\mathbb{C}$-valued functions in G.

A function $f \in C_1(\Omega)$ is said to be (left) monogenic in Ω if $\partial_x f = 0$ in Ω, where

$$\partial_x f = \sum_{i,A} e_i e_A \frac{\partial f_A}{\partial_{x_i}}.$$

On $L_2(\sum)$, the inner product is defined by

$$(f,g)_{\sum} = \int_{\sum} \overline{f(y)} g(y) dS(y).$$

Notice that $(f,g)_{\sum}$ is $\mathcal{A}$-valued and that if we put

$$[f,g]_{\sum} = \left[(f,g)_{\sum} \right]_0,$$

the scalar part of $(f,g)_{\sum}$, then $[\ ,\ ]_{\sum}$ is a classical inner product on $L_2(\sum)$ with

$$\begin{aligned} \| f \|_{\sum}^2 &= \int_{\sum} [\bar{f} f]_0 \, dS \\ &= \sum_A \int_{\sum} |f_A|^2 \, dS. \end{aligned}$$

The Cauchy kernel $C_x(y)$ in Ω is defined by

$$C_x(y) = \nu(y) E(y-x), \ x \in \Omega, \ y \in \Sigma, \tag{2.1}$$

and the associated Cauchy transform $\mathcal{C}f$ in $L_2(\sum)$ is given by

$$\begin{aligned} \mathcal{C}f(x) &= (C_x, f)_{\sum} \\ &= \frac{1}{A_{m+1}} \int_{\sum} \frac{x-y}{|x-y|^{m+1}} \, \nu(y) f(y) dS(y). \end{aligned} \tag{2.2}$$

Clearly, $\mathcal{C}f$ is monogenic in $\mathbb{R}^{m+1} \backslash \sum$.
Let $H^2(\Omega)$ be the Hardy space of $\mathcal{A}$-valued monogenic functions in Ω having non-tangential $L_2(\sum)$-boundary values and let $H^2(\sum)$ be the Hardy space on $\sum$ consisting of the boundary values of elements in $H^2(\Omega)$. Then $H^2(\sum)$ is a closed subspace of $L_2(\sum)$.

We have:

(i) $\mathcal{C}$ maps $L_2(\sum)$ onto $H^2(\Omega)$ and for $f \in L_2(\sum)$, the boundary value $\mathcal{C}f$ satisfies

the Plemelj-Sokhotzki formula

$$\mathcal{C}f = \frac{1}{2}(f + Hf) \tag{2.3}$$

where for a.e. $x \in \sum$,

$$Hf(x) = \frac{2}{A_{m+1}} \text{ P.V.} \int_{\sum} \frac{x-y}{|x-y|^{m+1}} \, \nu(y) f(y) dS(y). \tag{2.4}$$

Hereby H, called the Hilbert transform on $L_2(\sum)$, is a bounded linear operator satisfying $H^2 = \mathbf{1}$.
Straightforward computations show that the adjoint H^* of H is given by

$$H^* = \nu H \nu. \tag{2.5}$$

Moreover, for $f \in L_2(\sum)$, we have that $Hf = f$ if and only if $f \in H^2(\sum)$.

(ii) $\mathcal{C}$ is a bounded skew projection operator on $L_2(\sum)$, called the Hardy

projection. It maps $L_2(\sum)$ onto $H^2(\sum)$.

(iii) $L_2(\sum)$ admits the orthogonal decomposition

$$L_2(\Sigma) \;=\; H^2(\Sigma) \oplus \; \nu H^2(\Sigma). \tag{2.6}$$

The orthogonal projection operator $\mathbb{P}$ of $L_2(\sum)$ onto $H^2(\sum)$ - also called the Szegö projection - may be monogenically extended to $H^2(\Omega)$ by

$$Sf(x) = (S_x(y), f)_{\sum}$$

where $S_x(y), (x,y) \in \Omega \times \sum$, is the Szegö kernel for $H^2(\sum)$.
$S_x(y)$ is symmetric, i.e. $\overline{S_x(y)} = S_y(x)$ and it is the reproducing kernel for $H^2(\sum)$.

(iv) The projection operators $\mathbb{P}$ and $\mathcal{C}$ are related by the Kerzman-Stein formula

$$\mathbb{P}(\mathbf{1} + \mathbb{A}) = \mathcal{C}, \tag{2.7}$$

or equivalently by

$$P = \mathcal{C} - \mathbb{A} + \mathbb{A}\mathbb{P}.$$

Hereby $\mathbb{A} = \mathcal{C} - \mathcal{C}^*$ is the Kerzman-Stein operator in $L_2(\sum)$.

Notice that, as $\mathbb{A} = \dfrac{1}{2}(H - \nu H \nu)$, its kernel $A(x,y)$ is given by

$$A(x,y) = \frac{1}{A_{m+1}} \left(\nu(y) \frac{x-y}{|x-y|^{m+1}} + \frac{x-y}{|x-y|^{m+1}} \nu(x) \right). \tag{2.8}$$

For the properties (i) - (iv) we refer to [6], [7] and [2].

(v) If Ω is the ball $\overset{\circ}{B}(a,R)$ with center a and radius R, then its Szegö kernel is

given by

$$S_x(y) = \frac{1}{A_{m+1}} \frac{y-a}{R} \frac{x-y}{|x-y|^{m+1}}, \tag{2.9}$$

while for the upper half space $\mathbb{R}^{m+1}_+$,

$$S_x(y) = \frac{1}{A_{m+1}} \quad \frac{-e_m x - y e_m}{|x e_m + e_m y|^{m+1}}. \tag{2.10}$$

For these results we refer e.g. to [6].
It is easily seen that in both cases

(v.1) $S_x(y) = C_x(y)$, i.e. the Szegö and Cauchy kernels coincide;

(v.2) $\mathbb{P} = \mathcal{C}$, i.e. the Szegö and Hardy projections coincide; (2.11)

(v.3) $H = H^*$, i.e. the Hilbert transform H is unitary.

(vi) An algebraic decomposition of $L_2(\sum)$ may be obtained as follows.

Call α and β the functions defined on $\sum$ by

$$\alpha(x) = \frac{1}{2}\ (1 + i\nu(x))$$

and

$$\beta(x) = \frac{1}{2}\ (1 - i\nu(x))\,.$$

Then clearly for each $x \in \sum$, $\alpha(x)$ and $\beta(x)$ are hermitian orthogonal primitive idempotents in $\mathbb{C}_{m+1}$, i.e.

$$\alpha^2 = \alpha\ ,\ \beta^2 = \beta;$$
$$\alpha\beta = 0;$$
$$\bar{\alpha} = \alpha\ ;\ \bar{\beta} = \beta.$$

Moreover

$$\alpha + \beta = 1.$$

Consequently, for any $f, g \in L_2(\sum)$,

$$(\alpha f, \beta g)_{\sum} = 0,$$

whence the following orthogonal decomposition is obtained:

$$L_2(\Sigma) = \alpha L_2(\Sigma) \oplus \beta L_2(\Sigma). \tag{2.12}$$

The following theorem gives characterizations for the Hilbert transform H to be unitary.

Theorem 2.1. Let Ω be a bounded open domain in $\mathbb{R}^{m+1}$ with C_∞-boundary $\sum$. Then the following are equivalent:

(i) $\alpha H(\alpha f) = 0$ and $\beta H(\beta f) = 0$ for all $f \in L_2(\sum)$.

(ii) $H(\alpha f) \ = \ \beta f$ for all $f \in H^2(\sum)$.

(iii) $H(\beta f) = \alpha f$ for all $f \in H^2(\sum)$.

(iv) $H(\nu f) = -\nu f$ for all $f \in H^2(\sum)$.

(v) H is unitary.

(vi) $\mathbb{A} = 0$.

(vii) Ω is a ball.

(viii) $S_x(y) = C_x(y)$, i.e. the Szegö and Cauchy kernels coincide.

Proof. Straightforward arguments lead to the equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ ... $\Leftrightarrow$ (vi).

(vi) $\Rightarrow$ (vii).
¿From $\mathbb{A} = 0$ and the expression (2.8) of $A(x, y)$, it follows that for any elements $x, y \in \sum$ with $x \neq y$, $\nu(y)(y - x) = (x - y)\nu(x)$.
In view of [16] Lemma 12, Ω is a ball.

(vii) $\Rightarrow$ (vi).
In the case of a ball, H is unitary (see (2.11),(v.3)), whence $H = \nu H \nu$ and so $\mathbb{A} = 0$.

(vii) $\Rightarrow$ (viii).
For a ball, $S_x(y) = C_x(y)$ (see (2.11),(v.1)).

(viii) $\Rightarrow$ (vii).
If $S_x(y) = C_x(y)$, then $\mathbb{P} = \mathcal{C}$ whence $\mathcal{C} = \mathcal{C}^*$ and so $\mathbb{A} = 0$. Consequently in view of (vi) $\Leftrightarrow$ (vii), Ω is a ball. $\square$

Notice in particular that Theorem 2.1 tells us that a ball is the only bounded open domain Ω in $\mathbb{R}^{m+1}$ with smooth boundary $\sum$ such that the Hilbert transform H is unitary in $L_2(\sum)$. Clearly, by virtue of the Plemelj-Sokhotzki formula (2.3), H is unitary if and only if the Cauchy transform $\mathcal{C}$ is self-adjoint. For a simular result in an even more general context, we refer to [16]. See also [11] for the case of the plane.

3 The Hilbert transform on S^m

In this section, we take $\Omega = \overset{\circ}{B}(1)$ the unit ball in $\mathbb{R}^{m+1}$. In this case $\sum = S^m$ and at each point $\omega \in S^m$, $\nu(\omega) = \omega$.
The Hilbert transform on S^m thus reads:

$$Hf(\xi) = \frac{2}{A_{m+1}} \text{ P.V. } \int_{S^m} \frac{\xi - \omega}{|\xi - \omega|^{m+1}} \, \omega f(\omega) dS(\omega), \; f \in L_2(S^m). \tag{3.1}$$

It is a unitary operator with eigenspaces $H^2(S^m)$ and $\omega H^2(S^m)$ corresponding to the eigenvalues $+1$ and -1.
In [9] it was shown that $H^2(S^m)$ admits the orthogonal decomposition

$$H^2(S^m) = \sum_{k=0}^{\infty} \oplus \, \mathcal{M}^+(k)$$

whence, in view of (2.6), $L_2(S^m)$ is orthogonally decomposed into

$$L_2(S^m) = \sum_{k=0}^{\infty} \oplus \, \mathcal{M}^+(k) \oplus \; \omega \sum_{k=0}^{\infty} \oplus \, \mathcal{M}^+(k).$$

Hereby the elements of $\mathcal{M}^+(k)$ and $\mathcal{M}^-(k) = \omega\mathcal{M}^+(k), \; k \in \mathbb{N}$, are called spherical monogenics of degree k.
As a right module over $\mathcal{A}$, $\mathcal{M}^+(k)$ has dimension $K(m;k)$ with

$$K(m;k) = \frac{(k+m-1)!}{k!(m-1)!}.$$

Choosing an orthonormal basis $\left(P_{k,i(k)}\right)_{i(k)=1}^{K(m;k)}$ of $\mathcal{M}^+(k)$ (see e.g. [9]) we obtain that if $u \in L_2(S^m)$ has the decomposition

$$u(\omega) = f_1(\omega) + \omega f_2(\omega) \; , \; f_1, f_2 \in H^2(S^m), \tag{3.2}$$

then for $j = 1, 2$,

$$f_j(\omega) = \sum_{k=0}^{\infty} \sum_{i(k)=1}^{K(m;k)} P_{k,i(k)}(\omega) a_{k,i(k)}^{(j)}, \tag{3.3}$$

where for $k \in \mathbb{N}$,

$$a_{k,i(k)}^{(1)} = \left(P_{k,i(k)}, u\right)_{S^m}$$

and

$$a_{k,i(k)}^{(2)} = \left(\omega P_{k,i(k)}, u\right)_{S^m}.$$

Furthermore, as H is unitary,

$$Hu = f_1 - \omega f_2. \tag{3.4}$$

Remarks.

(1) In [9], the orthogonal projections of $L_2(S^m)$ onto $H^2(S^m)$ and $H^2(S^m)^\perp$ were denoted by Π^+ and Π^- and the operator $H_\xi = \Pi^- - \Pi^+$, called the Hilbert-Riesz transform on $L_2(S^m)$, was introduced. It is clear that $H_\xi = -H$, thus showing that H_ξ is in fact a singular integral operator.
Let us also recall that if $\triangle_\xi$ and Γ_ξ denote, respectively, the Laplace-Beltrami operator and the spherical Dirac operator on $L_2(S^m)$, then

$$\sqrt{\left(\frac{m-1}{2}\right)^2 \mathbf{1} - \triangle_\xi} \;=\; H_\xi \left(\Gamma_\xi - \frac{m-1}{2}\,\mathbf{1}\right).$$

(2) From (3.2) and (3.3) we get straightforwardly - and this by replacing $\omega \in S^m$ by $x = r\omega \in \overset{\circ}{B}(1)$ $(0 \le r < 1)$ - that the unique solution to the Dirichlet problem

$$\begin{cases} \triangle U = 0 & \text{in } \overset{\circ}{B}(1) \\ U\,|_{S^m} = u, & u \in L_2(S^m) \end{cases}$$

is given by

$$U(x) \;=\; F_1(x) + xF_2(x). \tag{3.5}$$

Hereby F_1 and F_2, both belonging to $H^2(\overset{\circ}{B}(1))$, extend monogenically f_1 and f_2 to $\overset{\circ}{B}(1)$.
For the sake of completeness, let us recall that $H^2(\overset{\circ}{B}(1))$ may be characterized as being the space of those elements F monogenic in $\overset{\circ}{B}(1)$ for which

$$\sup_{r<1} \int_{S^m} \left|F(r\omega)\right|^2 dS(\omega) < +\infty\ .$$

Notice too that (3.5) was already obtained in [8] and this by using the decomposition of the Poisson kernel $\mathcal{P}_x(\omega)$ for $\overset{\circ}{B}(1)$ in terms of the Szegö kernel, namely

$$\mathcal{P}_x(\omega) \;=\; S_x(\omega) + \omega S_x(\omega)\bar{x}\ ,\ (x,\omega) \in\ \overset{\circ}{B}(1)) \times S^m.$$

For a similar result, see also [2].

4 The Hilbert transform on $\mathbb{R}^m$

Take $\Omega = \mathbb{R}^{m+1}_+ = \{\mathbf{x} = (x_0, \cdots, x_m) = (x, x_m) :\ x \in \mathbb{R}^m,\ x_m > 0\}$, its boundary being $\mathbb{R}^m = \{\mathbf{x} \in \mathbb{R}^{m+1} :\ x_m = 0\}$.
At each point $y \in \mathbb{R}^m,\ \nu(y) = \bar{e}_m$ and so the Hilbert transform on $\mathbb{R}^m$ is given for a.e. $x \in \mathbb{R}^m$ by

$$Hf(x) = \frac{2}{A_{m+1}}\ \text{P.V.} \int_{\mathbb{R}^m} \frac{x-y}{|x-y|^{m+1}}\ \bar{e}_m f(y)dy. \tag{4.1}$$

It is a unitary operator with eigenspaces $H^2(\mathbb{R}^m)$ and $\bar{e}_m H^2(\mathbb{R}^m)$, corresponding to the eigenvalues $+1$ and -1.
In [10] the operator $\widetilde{\mathcal{H}}$ was introduced with

$$\widetilde{\mathcal{H}} = \sum_{j=0}^{m-1} e_j \, R_j, \tag{4.2}$$

$R_j, j = 0, \cdots, m-1$, being the j-th Riesz transform:

$$R_j f(x) = \frac{2}{A_{m+1}} \text{ P.V.} \int_{\mathbb{R}^m} \frac{x_j - y_j}{|x-y|^{m+1}} f(y) dy.$$

From (4.1) and (4.2) it follows that

$$H = e_m \, \widetilde{\mathcal{H}}. \tag{4.3}$$

Introducing spherical coordinates $x = r\omega$ in $\mathbb{R}^m$ where $r = |x|$ and $\omega \in S^{m-1}$, we have that on $\mathcal{S}(\mathbb{R}^m)$

$$\begin{aligned} \widetilde{\mathcal{H}}\varphi(x) &= \left(- \text{ P.V.} \frac{\bar{\omega}}{r^m} * \varphi \right)(x) \\ &= \lim_{\varepsilon \to 0+} \int_{\mathbb{R}^m \setminus B(\varepsilon)} \frac{x-y}{|x-y|} \, \frac{\varphi(y)}{|x-y|^m} dy. \end{aligned}$$

Hereby P.V. $\dfrac{\bar{\omega}}{r^m}$ is the principal value kernel in $\mathbb{R}^m$ which generalizes in a natural way to Euclidean space the principal value kernel P.V. $\dfrac{1}{x}$ defined on $\mathbb{R}$.
For more generalized principal value distributions and associated convolution operators in Clifford analysis, we refer to [4] and [5].

Remarks.

(1) (See also [10]). The Clifford algebra $\mathbb{R}_{0,m+1}$ admits the splitting

$$\mathbb{R}_{0,m+1} = \mathbb{R}_{0,m} \oplus e_m \mathbb{R}_{0,m} \tag{4.4}$$

where $\mathbb{R}_{0,m}$ is the universal Clifford algebra constructed over the quadratic vector space $\mathbb{R}^{0,m}$ with orthogonal basis $(e_0, e_1, ..., e_{m-1})$.
If $f \in L_2(\mathbb{R}^m; \mathbb{R}_{0,m})$, then it follows from (4.2) that $\widetilde{\mathcal{H}}$ f is also $\mathbb{R}_{0,m}$-valued. Consequently, if for $f \in L_2(\mathbb{R}^m; \mathbb{R}_{0,m+1})$, its splitting in terms of (4.4) reads

$$f = U + e_m \, V$$

with $U, V \in L_2(\mathbb{R}^m; \mathbb{R}_{0,m})$, we have in view of (4.3) that

$$f \in H^2(\mathbb{R}^m) \Longleftrightarrow Hf = f \Longleftrightarrow \left\{ \begin{array}{rcl} \widetilde{\mathcal{H}}\, U &=& V \\ \widetilde{\mathcal{H}}\, V &=& U \end{array} \right. . \tag{4.5}$$

But, as $\widetilde{\mathcal{H}}^2 = \mathbf{1}$, the condition (4.5) reduces to

$$f \in H^2(\mathbb{R}^m) \Longleftrightarrow \widetilde{\mathcal{H}}U \;=\; V.$$

(2) The Poisson kernel $\mathcal{P}_{\mathbf{X}}(y)$ on $\mathbb{R}^{m+1}_+$ splits into

$$\mathcal{P}_{\mathbf{X}}(y) \;=\; S_{\mathbf{X}}(y) + \bar{e}_m S_{\mathbf{X}}(y) e_m \tag{4.6}$$

where $S_{\mathbf{X}}(y)$ is the Szegö kernel for $\mathbb{R}^{m+1}_+$ (see also (2.10)).
Using the orthogonal decomposition

$$L_2(\mathbb{R}^m) \;=\; H^2(\mathbb{R}^m) \oplus \bar{e}_m H^2(\mathbb{R}^m),$$

it readily follows from (4.6) that a function $f \in L_2(\mathbb{R}^m)$ may be written as

$$f \;=\; \mathbb{P}f + \bar{e}_m \mathbb{P}(e_m f),$$

where $\mathbb{P}$ is the Szegö projection from $L_2(\mathbb{R}^m)$ onto $H^2(\mathbb{R}^m)$.
Call $\mathcal{H}^2(\mathbb{R}^{m+1}_+)$ (resp. $H^2(\mathbb{R}^{m+1}_+)$) the space of harmonic (resp. monogenic) functions in $\mathbb{R}^{m+1}_+$ having non-tangential boundary values in $L_2(\mathbb{R}^m)$. Then (see [15] and [10]), F belongs to either of these spaces iff

$$\sup_{y_m>0} \int_{\mathbb{R}^m} |F(y, y_m)|^2 dy \;< +\infty.$$

As the Szegö integral operator S associated with the Szegö kernel $S_{\mathbf{X}}(y)$ maps $H^2(\mathbb{R}^m)$ onto $H^2(\mathbb{R}^{m+1}_+)$ and the Poisson integral operator $\mathcal{P}$ maps $L_2(\mathbb{R}^m)$ onto $\mathcal{H}^2(\mathbb{R}^{m+1}_+)$, we obtain the decomposition

$$\mathcal{H}^2(\mathbb{R}^{m+1}_+) \;=\; H^2(\mathbb{R}^{m+1}_+) \oplus \bar{e}_m H^2(\mathbb{R}^{m+1}_+).$$

Hence, for any $U \in \mathcal{H}^2(\mathbb{R}^{m+1}_+)$, there exists a pair of functions (F_1, F_2) in $H^2(\mathbb{R}^{m+1}_+)$ such that

$$U \;=\; F_1 + \bar{e}_m F_2.$$

For a similar result, see also [2].

5 Classical results in the plane revisited

In this section it is shown how the Clifford analysis approach, as set up in the foregoing sections, includes classical results when considering the case $\mathbb{R}^2$, i.e. when taking $m = 1$. For a standard approach to the Cauchy and Hilbert transforms in the plane, we refer to [1].

5.1 Decompositions of $H^2(\Omega)$ and $L_2(\sum)$

Let $\Omega \subset \mathbb{R}^2$ be open and bounded with C_∞-boundary $\sum$ or let Ω be $\mathbb{R}^2_+ = \{x = (x_0, x_1) : x_1 > 0\}$ with boundary $\mathbb{R} = \{x \in \mathbb{R}^2 : \ x_1 = 0\}$.
Put $\varepsilon_1 = \ \bar{e}_0 e_1$. Then the even subalgebra $\mathbb{R}^+_{0,2}$ of $\mathbb{R}_{0,2}$ is given by

$$\mathbb{R}^+_{0,2} \ = \ \mathbb{R} \oplus \varepsilon_1 \mathbb{R}.$$

It gives rise to the decomposition

$$\mathbb{R}_{0,2} \ = \ \mathbb{R}^+_{0,2} \oplus \mathbb{R}^+_{0,2}\bar{e}_0. \tag{5.1}$$

Notice that by identifying $\varepsilon_1 = \bar{e}_0 e_1$ with the imaginary unit i in $\mathbb{C}$, $\mathbb{R}^+_{0,2}$ is isomorphic to $\mathbb{C}$.
In what follows, only $\mathbb{R}_{0,2}$-valued functions will be considered.
In view of (5.1), if F is defined on some subset G of $\mathbb{R}^2$, then F may be written as

$$F(x) \ = \ U(x) + V(x)\bar{e}_0, \ \ x \in G, \tag{5.2}$$

where U and V are $\mathbb{R}^+_{0,2}$-valued on G.
If F is a monogenic function in Ω, i.e. $\partial_x F = 0$ in Ω, then as

$$\begin{aligned} \partial_x F = 0 \ \Longleftrightarrow \ \bar{e}_0 \partial_x F \ &= \ 0 \\ \Longleftrightarrow \ D_x F \ &= \ 0 \end{aligned}$$

where $D_x = \partial_{x_0} + \varepsilon_1 \partial_{x_1}$ is the Weyl or Cauchy-Riemann operator in $\mathbb{R}^2$, we obtain by virtue of (5.1) that

$$\partial_x F = 0 \Longleftrightarrow \left\{ \begin{array}{lcl} D_x U & = & 0 \\ D_x V & = & 0 \end{array} \right. \quad \text{in } \Omega.$$

Notice hereby that for $g = u + \varepsilon_1 v, \mathbb{R}^+_{0,2}$-valued in Ω, $D_x g = 0$ in Ω is equivalent to saying that the $\mathbb{C}$-valued function $g^* = u + iv$ is holomorphic in Ω.
We may thus conclude that a monogenic $\mathbb{R}_{0,2}$-valued function F in Ω may be identified with a pair (g_1, g_2) of holomorphic functions in Ω and vice-versa.
Notice too that

$$H^2(\Omega) \ = \ H^2(\Omega; \mathbb{R}^+_{0,2}) \oplus H^2(\Omega; \mathbb{R}^+_{0,2})\bar{e}_0$$

and that the space of boundary values of elements in $H^2(\Omega; \mathbb{R}^+_{0,2})$ is $H^2(\sum; \mathbb{R}^+_{0,2})$.
Now let $f = u + v\bar{e}_0 \in L_2(\sum)$, i.e. $u, v \in L_2(\sum; \mathbb{R}^+_{0,2})$.
Taking the Cauchy and Hilbert transforms of f we find that, as $(x - y)\nu(y)$ is $\mathbb{R}^+_{0,2}$-valued, $\mathcal{C}u, \mathcal{C}v, Hu$ and Hv are $\mathbb{R}^+_{0,2}$-valued too.
Consequently, in terms of the decomposition (5.1),

$$\mathcal{C}f(x) \ = \ \mathcal{C}u(x) + \mathcal{C}v(x)\bar{e}_0, \ \ x \in \Omega,$$

and (5.3)

$$Hf(x) \ = \ Hu(x) + Hv(x)\bar{e}_0, \ \ x \in \Sigma.$$

Notice hereby that on $L_2(\sum;\mathbb{R}^+_{0,2})$, the inner product

$$(f,g)_{\sum} = \int_{\sum} \bar{f} g ds$$

is also $\mathbb{R}^+_{0,2}$-valued.
Let us now have a closer look at the Kerzman-Stein decomposition

$$L_2(\Sigma) = H^2(\Sigma) \oplus \nu H^2(\Sigma). \tag{5.4}$$

Then it is easily verified that, by mixing up (5.1) and (5.4), we obtain:

$$\begin{aligned} L_2(\Sigma) &= H^2(\Sigma;\mathbb{R}^+_{0,2}) \oplus \nu H^2(\Sigma;\mathbb{R}^+_{0,2})\bar{e}_0 \\ &\oplus \left(H^2(\Sigma;\mathbb{R}^+_{0,2}) \oplus \nu H^2(\Sigma;\mathbb{R}^+_{0,2})\bar{e}_0\right)\bar{e}_0. \end{aligned} \tag{5.5}$$

In particular, $L_2(\sum;\mathbb{R}^+_{0,2})$ is orthogonally decomposed into

$$L_2(\Sigma;\mathbb{R}^+_{0,2}) = H^2(\Sigma;\mathbb{R}^+_{0,2}) \oplus \nu H^2(\Sigma;\mathbb{R}^+_{0,2})\bar{e}_0. \tag{5.6}$$

So, in order to describe $L_2(\sum)$, it suffices to determine $H^2(\sum;\mathbb{R}^+_{0,2})$.

Remark. Let $\Omega \subset \mathbb{R}^2$ be open and bounded with C_∞-boundary $\sum$. By virtue of (2.2), the Cauchy transform on $L_2(\sum;\mathbb{R}^+_{0,2})$ is given by

$$\begin{aligned} \mathcal{C}f(x) &= (C_x, f)_{\sum} \\ &= \frac{1}{2\pi}\int_{\sum} \frac{x-y}{|x-y|^2}\nu(y)f(y)ds(y). \end{aligned} \tag{5.7}$$

As we have already observed, $\mathcal{C}f$ is $\mathbb{R}^+_{0,2}$-valued and monogenic in Ω and, moreover, monogenic $\mathbb{R}^+_{0,2}$-valued functions in Ω may be identified with holomorphic functions in Ω.
Usually, if in the complex plane $\sum$ is parametrized by $z(t), t \in [0,1]$, and $T(z) = \dfrac{z'(t)}{|z'(t)|}$ denotes the unit tangent vector at $z(t) \in \sum$, the Cauchy transform $\mathcal{C}f$ of $f \in L_2(\sum;\mathbb{C})$ is defined by (see [1])

$$\mathcal{C}f(u) = \frac{1}{2\pi i}\int_{\sum} \frac{1}{z-u} T(z) f(z) ds. \tag{5.8}$$

But, as at $z \in \sum$, the outward pointing unit normal $n(z)$ is given by

$$n(z) = -iT(z),$$

the expression (5.8) also reads:

$$\mathcal{C}f(u) = \frac{1}{2\pi}\int_{\sum} \frac{\overline{z-u}}{|z-u|^2} n(z) f(z) ds. \tag{5.9}$$

Obviously, the expressions (5.7) and (5.9) are similar.

5.2 The case of the unit disc

Let $\Omega = \overset{\circ}{B}(1)$ be the open unit disc in $\mathbb{R}^2$ with boundary $\sum = S^1$.
From (3.1) we obtain for a.e. $\xi \in S^1$:

$$Hf(\xi) \;=\; \frac{1}{\pi}\text{P.V.}\int_{S^1} \frac{\xi-\omega}{|\xi-\omega|^2}\omega f(\omega)ds(\omega).$$

As has been observed in § 5.1, it suffices to characterize $H^2(S^1;\mathbb{R}^+_{0,2})$ in order to describe $L_2(S^1;\mathbb{R}^+_{0,2})$ and $L_2(S^1)$. Moreover, through the identifications $\mathbb{R}^+_{0,2} \cong \mathbb{C}$ and $H^2(\Omega;\mathbb{R}^+_{0,2}) \cong H^2(\Omega,\mathbb{C}), H^2(S^1;\mathbb{R}^+_{0,2})$ is nothing else but the classical Hardy space $H^2(S^1;\mathbb{C})$.

An orthonormal basis of $H^2(S^1;\mathbb{R}^+_{0,2})$ is thus given by

$$\left(\frac{1}{\sqrt{2\pi}}e^{\varepsilon_1 k\theta} : k \in \mathbb{N}, \theta \in [0,2\pi[\right), \tag{5.10}$$

whence by means of (5.6)

$$\left(\frac{1}{\sqrt{2\pi}}\nu e^{\varepsilon_1 k\theta}\bar{e}_0 : k \in \mathbb{N}, \theta \in [0,2\pi[\right)$$

is an orthonormal basis of $\nu H^2(\sum;\mathbb{R}^+_{0,2})\bar{e}_0$.
As for $\omega \in S^1$,

$$\nu(\omega) \;=\; \omega \;=\; e_0 e^{\varepsilon_1\theta},$$

we obtain that for each $k \in \mathbb{N}$,

$$\nu e^{\varepsilon_1 k\theta}\bar{e}_0 \;=\; e^{-\varepsilon_1(k+1)\theta}.$$

Consequently, the orthonormal basis

$$\left(\frac{1}{\sqrt{2\pi}}e^{-\varepsilon_1(k+1)\theta} : k \in \mathbb{N}, \theta \in [0,2\pi[\right) \tag{5.11}$$

of $H^2(S^1;\mathbb{R}^+_{0,2})^\perp \;=\; \omega H^2(S^1;\mathbb{R}^+_{0,2})\bar{e}_0$ is obtained.

Furthermore, as for $f \in H^2(S^1;\mathbb{R}^+_{0,2}), Hf = f$ and for $f \in H^2(S^1;\mathbb{R}^+_{0,2})^\perp, Hf = -f$, we have in particular that for each $k \in \mathbb{N}$,

$$H\left(e^{\varepsilon_1 k\theta}\right) \;=\; e^{\varepsilon_1 k\theta}$$

and (5.12)

$$H\left(e^{-\varepsilon_1(k+1)\theta}\right) \;=\; -e^{-\varepsilon_1(k+1)\theta}.$$

Remarks.

(1) In [14] the periodic Hilbert transform $\widetilde{H}$ on $L_2(S^1;\mathbb{C})$ was introduced as follows:

$$\widetilde{H}\left(e^{in\theta}\right) = \frac{\text{sgn } n}{i} e^{in\theta}, n \in \mathbb{Z}\backslash\{0\},$$

and

$$\widetilde{H}(1) = 0.$$

It was extended by linearity to a bounded operator on $L_2\ (S^1;\mathbb{C})$.
Identifying i with $\varepsilon_1 = \bar{e}_0 e_1$ and denoting by $\mathbb{P}_0$ the orthogonal projection of L_2 $(S^1;\mathbb{R}^+_{0,2})$ onto its one dimensional subspace of constants, then clearly

$$H = \varepsilon_1 \tilde{H} + \mathbb{P}_0.$$

(2) Putting $z = x_0 + \varepsilon_1 x_1$, then for each $k \in \mathbb{N}$,

$$z^k|_{S^1} = e^{\varepsilon_1 k\theta} \text{ and } \bar{z}^{k+1}|_{S^1} = e^{-\varepsilon_1(k+1)\theta} = \nu e^{\varepsilon_1 k\theta}\bar{e}_0.$$

Consequently, the harmonic extensions of $e^{\varepsilon_1 k\theta}$ and $\nu e^{\varepsilon_1 k\theta}\bar{e}_0$ to $\overset{\circ}{B}(1)$ are given by z^k and $\bar{z}^{k+1} = xz^k\bar{e}_0$. In its turn, this implies that the (unique) harmonic extension U of $u \in L_2(S^1;\mathbb{R}^+_{0,2})$ to $\overset{\circ}{B}(1)$ may be written as $U(x) = F_1(x) + F_2(x)$ where F_1 and F_2 have series expansions into z and $\bar{z}$ respectively. In terms of holomorphy, this means that F_1 is holomorphic and F_2 is anti-holomorphic in $\overset{\circ}{B}(1)$. Although this result is standard, we wish to give some supplementary comments on it, in particular concerning its interpretation in Clifford analysis terms.
Let $u \in L_2(S^1;\mathbb{R}^+_{0,2})$ admit the decomposition (see (5.6)):

$$u = w_0 + \nu w_1 \bar{e}_0$$

where $w_0, w_1 \in H^2(S^1;\mathbb{R}^+_{0,2})$.
The (unique) harmonic extension $U \in \mathcal{H}^2\left(\overset{\circ}{B}(1);\mathbb{R}^+_{0,2}\right)$ of u to $\overset{\circ}{B}(1)$ then reads:

$$U(x) = W_0(x) + xW_1(x)\bar{e}_0 \tag{5.13}$$

where $W_0, W_1 \in H^2\left(\overset{\circ}{B}(1);\mathbb{R}^+_{0,2}\right)$.
As W_1 is $\mathbb{R}^+_{0,2}$-valued and monogenic,

$$D_x W_1 = W_1 D_x = 0 \text{ in } \overset{\circ}{B}(1)$$

whence

$$W_1 \bar{e}_0 \partial_x = W_1 D_x = 0 \text{ in } \overset{\circ}{B}(1).$$

Moreover, as $xW_1 D_x = 0$, we finally obtain that

$$\left(x\, W_1(x)\bar{e}_0\right)\partial_x = \left(x\, W_1(x)\right) D_x = 0 \text{ in } \overset{\circ}{B}(1),$$

i.e. $xW_1(x)\bar{e}_0$ is right monogenic in $\overset{\circ}{B}(1)$
Now observe that for any $f \in C_1(\Omega; \mathbb{R}^+_{0,2})$,

$$f\partial_x \;=\; 0 \Longleftrightarrow \partial_x \bar{f} \;=\; 0 \Longleftrightarrow D_x \bar{f} \;=\; 0.$$

In terms of holomorphy, this means that f is anti-holomorphic in Ω.
Therefore, the expression (5.13) is similar to the one obtained in the complex plane (see [1], Theorem 7.4). It represents the solution to the Dirichlet problem in the unit disc as a sum of a holomorphic and an anti-holomorphic function, the latter vanishing at $x = 0$.

5.3 The case of the upper half plane

Let $\Omega = \mathbb{R}^2_+ = \{(x_0, x_1) : x_1 > 0\}$ be the upper half plane in $\mathbb{R}^2$ with boundary $\sum = \mathbb{R}$.
¿From (4.1) we get that for $f \in L_2(\mathbb{R})$,

$$\begin{aligned} Hf(x_0) &= \frac{1}{\pi}\text{P.V.}\int_{-\infty}^{+\infty} \frac{x_0 - y_0}{|x_0 - y_0|^2} e_0\bar{e}_1 f(y_0)dy_0 \\ &= \varepsilon_1 \frac{1}{\pi}\text{ P.V. }\int_{-\infty}^{+\infty} \frac{1}{x_0 - y_0} f(y_0)dy_0 \\ &= \varepsilon_1 \mathcal{H}f(x_0) \\ &= \varepsilon_1 \left(\text{P.V.}\frac{1}{y_0} * f\right)(x_0) \end{aligned} \tag{5.14}$$

where $\mathcal{H}$ is the standard Hilbert transform on $L_2(\mathbb{R})$.
It follows that if f is $\mathbb{R}$-valued, then Hf is $\varepsilon_1\mathbb{R}$-valued.
Hence, for $f \in L_2(\mathbb{R}; \mathbb{R}^+_{0,2})$ with $f = u + \varepsilon_1 v$, u and v being $\mathbb{R}$-valued, we have

$$Hf \;=\; f \Longleftrightarrow \begin{cases} Hu &= \varepsilon_1 v \\ H(\varepsilon_1 v) &= u \end{cases}. \tag{5.15}$$

But, as $H^2 = \mathbf{1}$, (5.15) reduces to

$$Hf \;=\; f \Longleftrightarrow Hu \;=\; \varepsilon_1 v. \tag{5.16}$$

In view of (5.14), (5.16) is equivalent with $\mathcal{H}u = v$, thus reobtaining a classical result for the Hilbert transform on the real line.

Remarks.

(1) Let $u \in L_2(\mathbb{R})$ be real valued and put

$$f \;=\; u + Hu. \tag{5.17}$$

Then $f \in L_2(\mathbb{R}; \mathbb{R}^+_{0,2})$ and $Hf = f$, i.e. $f \in H^2(\mathbb{R}; \mathbb{R}^+_{0,2})$, whence by means of (5.17) all $\mathbb{R} \oplus \varepsilon_1\mathbb{R}$-valued "analytic signals" are obtained.

(2) Consider on $\mathbb{R}$ the $\mathbb{C}_2^+$-valued functions

$$\sigma_\pm(x_0) \;=\; \frac{1}{2}(1 \pm i\varepsilon_1 \;\mathrm{sgn}\; x_0)$$

where

$$\mathbb{C}_2^+ \;=\; \mathbb{R}_{0,2}^+ \oplus_{\mathbb{R}} \mathbb{C}$$

is the even subalgebra of $\mathbb{C}_2$.
Then on $\mathbb{R}$

$$\begin{array}{lcl} \sigma_\pm^2 & = & \sigma_\pm; \\ \overline{\sigma_\pm} & = & \sigma_\pm; \\ \sigma_\pm\sigma_\mp & = & 0; \\ \sigma_+ + \sigma_- & = & 1, \end{array}$$

i.e. σ_+ and σ_- are hermitian, mutually orthogonal primitive idempotents in $\mathbb{C}_2$.
Taking the Fourier transform $\mathcal{F}^\pm$ in (5.14), we find that for $f \in L_2(\mathbb{R}; \mathbb{R}_{0,2}^+)$,

$$\mathcal{F}^\pm Hf(x_0) \;=\; \pm i\varepsilon_1 \;\mathrm{sgn} x_0 \;\mathcal{F}^\pm f.$$

Hence for $f \in L_2(\mathbb{R}; \mathbb{R}_{0,2}^+)$,

$$Hf \;=\; f \Longleftrightarrow \sigma_\pm \mathcal{F}^\pm f \;=\; 0.$$

This result should be compared with the classical one obtained for $\mathbb{C}$-valued functions, stating that for $f \in L_2(\mathbb{R})$ are equivalent (see e.g. [10])

(i) $f \in H^2(\mathbb{R})$

(ii) $(\mathbf{1} - i\mathcal{H})f = 0$

(iii) $[\mathcal{F}^- f] \subset [0, +\infty[$.

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