Classical solutions for PDEs with nonautonomous past in L^p- Spaces

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Abstract

We study some partial differential equations with infinite delay which appear, for example, in models of population dynamics. Using the semigroup theory, we prove the existence of classical solutions of such equations.

1 Introduction

In [6] S. Brendle and R. Nagel introduced the equations

$$\frac{\partial}{\partial t}u(t,s) = \frac{\partial}{\partial s}u(t,s) + A(s)u(t,s), \quad s \leq 0,\ t \geq 0, \tag{1.1}$$

$$\frac{\partial}{\partial t}u(t,0) = Bu(t,0) + \Phi u(t,\cdot), \quad t \geq 0, \tag{1.2}$$

where $A(s)$ are (unbounded) operators on a Banach space X for which the associated nonautonomous Cauchy problem

$$(NCP) \quad \begin{cases} \dot{u}(t) & = -A(t)u(t),\ t \leq s \leq 0, \\ u(s) & = x \in X \end{cases}$$

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is well-posed, B is the generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ on X, and Φ, the delay operator, is a linear operator from a space of $X-$valued functions on $\mathbb{R}_-$ into X (see also [12], [13], [14] or [15]). They found mild solutions of the above equations by constructing a suitable semigroup $(T_{B,\Phi}(t))_{t\geq 0}$ on $C_0(\mathbb{R}_-, X)$, while we proved in [11] that, under suitable assumptions, their semigroup also yields classical solutions of (1.1) and (1.2), i.e., $u(t,s)$ is continuously differentiable and satisfies the two equations.

In this paper we study the above equations in $L^p(\mathbb{R}_-, X)$, since there are many applications in which such spaces are considered. For example, in some biological models u is the population density and so the natural space is L^1 (see [15], [21], [23], [24]). Moreover, in control theory one generally considers the space L^2 (see, e.g., [2]).

The steps to prove the main theorem are the same as [11, Theorem 4.4], but we need some different techniques to prove some lemmas and preliminary results useful for the proof of Theorem 4.6 (because, for example, bounded functions in $C_0(\mathbb{R}_-, X)$ may not be bounded in $L^p(\mathbb{R}_-, X)$).

In order to treat the L^p-case, we associate to (1.1) and (1.2) a delay equation with nonautonomous past

$$(NDE) \quad \begin{cases} \dot{u}(t) & = Bu(t) + \Phi \tilde{u}_t \\ u(0) & = x \in X \\ \tilde{u}_0 & = f \in L^p(\mathbb{R}_-, X), \end{cases}$$

(see [12] and [13]), where $\tilde{u}_t$ is the modified history function defined as

$$\tilde{u}_t(\tau) := \begin{cases} U(\tau, 0)u(t+\tau) & \text{for } t+\tau \geq 0 \geq \tau, \\ U(\tau, t+\tau)f(t+\tau) & \text{for } 0 \geq t+\tau \geq \tau. \end{cases}$$

Here $\mathcal{U} := (U(t,s))_{t\leq s\leq 0}$ is the evolution family solving (NCP) on regularity subspaces Y_s, i.e. there are dense subspaces Y_s of X such that the function $t \mapsto u(t) = U(t,s)x$ is a classical solution of (NCP) for $s \in \mathbb{R}_-$ and $x \in Y_s$ (see [20, Proposition 2.5]). A particular form of (NDE) has been studied by A. Bátkai and S. Piazzera (see, e.g., [3], [4] and [5]).

The basic idea in this paper is to consider a product space $\mathcal{E} := X \times L^p(\mathbb{R}_-, X)$ and to define an operator matrix which is related to (1.1) and (1.2). Using the Miyadera-Voigt perturbation theorem (see [10, Theorem III.3.14], [19] or [26]) we find a semigroup whose second coordinate gives, under suitable assumptions, classical solutions of (1.1) and (1.2). It must be noted that the L^p-case leads to an additional technical assumption on the delay operator Φ (see Assumption 3.9), which guarantees the existence of such a semigroup.

2 Motivations

Partial functional differential equations with autonomous past can be written in abstract form as

$$(DE) \quad \begin{cases} \dot{u}(t) & = Bu(t) + \Phi u_t, \ t \geq 0, \\ u(0) & = x \in X, \\ u_0 & = f \in L^p(\mathbb{R}_-, X), \end{cases}$$

where the function u takes values in a Banach space X and the history function $u_t : \mathbb{R}_- \to X$ is defined by

$$u_t(s) := u(t+s).$$

In [10, Section VI.6], it is shown that if $\Phi \in \mathcal{L}(C_0(\mathbb{R}_-, X), X)$, then a solution of (DE) is given by

$$u(t) := \begin{cases} f(t), & t \leq 0, \\ (T_{B,\Phi}(t)f)(0), & t \geq 0, \end{cases} \tag{2.1}$$

where $(T_{B,\Phi}(t))_{t\geq 0}$ is a suitable strongly continuous semigroup on $C_0(\mathbb{R}_-, X)$, which satisfies the following translation property

$$(T_{B,\Phi}(t)f)(s) := \begin{cases} f(s+t), & s+t \leq 0, \\ (T_{B,\Phi}(s+t)f)(0), & s+t \geq 0. \end{cases} \tag{2.2}$$

If, however, we define

$$u(t,s) := (T_{B,\Phi}(t)f)(s) \tag{2.3}$$

for $t \geq 0$ and $s \leq 0$, we obtain that u satisfies a *two-variable* version of (DE) of the form

$$\frac{\partial}{\partial t} u(t,s) = \frac{\partial}{\partial s} u(t,s), \qquad s \leq 0,\ t \geq 0, \tag{2.4}$$

$$\frac{\partial}{\partial t} u(t,0) = Bu(t,0) + \Phi u(t,\cdot), \quad t \geq 0. \tag{2.5}$$

The first equation says that the history function is shifted to left by $-t$, without being modified, while the second shows that, for values greater than $-t$, the function is due to the delay operator. In some cases the history function is not only shifted into the past, but it is also modified. For example, this occurs for population equations with diffusion (see, e.g., [15], [23] or [24]) and for genetic repression (see, e.g., [14] or [18]). Assume that this modification is governed by a backward evolution family $(U(t,s))_{t\leq s\leq 0}$(cf. [8]), i.e., a family of bounded linear operators on X satisfying

$$U(r,s)U(s,t) = U(r,s),\ r \leq s \leq t \leq 0, \tag{2.6}$$

$$U(t,t) = Id, \qquad t \leq 0, \tag{2.7}$$

and such that the mapping $(r,s) \mapsto U(r,s)$ is strongly continuous, and $(T_{B,\Phi}(t))_{t\geq 0}$ satisfies a modified translation property

$$(T_{B,\Phi}(t)f)(s) := \begin{cases} U(s+t,s)f(s+t), & s+t \leq 0, \\ U(0,s)(T_{B,\Phi}(s+t)f)(0), & s+t \geq 0, \end{cases} \tag{2.8}$$

for each $f \in C_0(\mathbb{R}_-, X)$. If we differentiate formally

$$u(t,s) := (T_{B,\Phi}(t)f)(s), \tag{2.9}$$

then u satisfies

$$\frac{\partial}{\partial t} u(t,s) = \frac{\partial}{\partial s} u(t,s) + A(s)u(t,s), \quad s \leq 0,\ t \geq 0, \tag{2.10}$$

for linear operators

$$A(s) := -\frac{\partial}{\partial r} U(r,s)|_{r=s}$$

on X. For this reasons, in [6], S. Brendle and R. Nagel replaced (2.4) with (2.10), and studied the combination of (1.1) and (1.2).

3 The Coordinates of the Semigroup

As a first step we fix the assumptions and review, following [12], the definitions and the results that will be used in the rest of the paper.

General assumptions 3.1. 1. The generator $(B, D(B))$ satisfies $D(B) \hookrightarrow Y_0$, where Y_0 is a regularity subspace as in the Introduction.

2. The (linear) delay operator $\Phi : C_0(\mathbb{R}_-, X) \cap L^p(\mathbb{R}_-, X) \subseteq D(\Phi) \to X$ is bounded with respect to $\|\cdot\|_p$ or $\|\cdot\|_\infty$.

3. The operators $(A(t), D(A(t)))_{t\in\mathbb{R}_-}$ are such that the function $s \mapsto A(s)(\epsilon_\lambda x)(s)$ belongs to $E := L^p(\mathbb{R}_-, X)$ for $x \in D(B)$, and the bounded linear operators $\epsilon_\lambda : X \to E$ are defined, for all $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega_0(\mathcal{U})$, by

$$(\epsilon_\lambda x)(s) := e^{\lambda s} U(s, 0)x,\ s \le 0,\ x \in X. \tag{3.1}$$

Here $\omega_0(\mathcal{U})$ denotes the growth bound of $\mathcal{U}$, i.e.

$$\omega_0(\mathcal{U}) := \inf\{\omega \in \mathbb{R} :\ \exists\, M_\omega \ge 1 \text{ with } \|U(t, s)\| \le M_\omega e^{\omega(s-t)} \text{ for } t \le s \le 0\}.$$

To use semigroup techniques we extend the evolution family $(U(t, s))_{t\le s\le 0}$ solving (NCP) to an evolution family $(\widetilde{U}(t, s))_{t\le s}$ on $\mathbb{R}$ in a trivial way (see [12, Definition 2.2.1]). On the space $\widetilde{E} := L^p(\mathbb{R}, X)$, we then define the corresponding **evolution semigroup** $(\widetilde{T}(t))_{t\ge 0}$ by

$$(\widetilde{T}(t)\widetilde{f})(s) := \widetilde{U}(s, s + t)\widetilde{f}(s + t)$$

for all $\widetilde{f} \in \widetilde{E}, s \in \mathbb{R}, t \ge 0$ (see also [7]).

We denote its generator by $(\widetilde{G}, D(\widetilde{G}))$. Remark that we did not assume any differentiability for $(\widetilde{U}(t, s))_{t\le s}$ and hence the precise description of the domain $D(\widetilde{G})$ is difficult. However, in [22, Proposition 2.1], the following important property of $D(\widetilde{G})$ is proved.

Lemma 3.2. *The domain $D(\widetilde{G})$ of the generator $\widetilde{G}$ of the evolution semigroup $(\widetilde{T}(t))_{t\ge 0}$ on $\widetilde{E}$ is a dense subspace of $C_0(\mathbb{R}, X)$.*

Since $(\widetilde{G}, D(\widetilde{G}))$ is a local operator (see [10, Proposition 2.3] and [22, Theorem 2.4]), we can restrict it to the space $E := L^p(\mathbb{R}_-, X)$ by the following definition.

Definition 3.3. Take

$$D(G) := \{\widetilde{f}_{|\mathbb{R}_-}\ :\ \widetilde{f} \in D(\widetilde{G})\}$$

and define

$$Gf := (\widetilde{G}\widetilde{f})_{|\mathbb{R}_-} \qquad \text{for } f = \widetilde{f}_{|\mathbb{R}_-} \in D(\widetilde{G}).$$

This operator G is not a generator on E. However, if we identify E with the subspace $\{f \in \widetilde{E} : f(s) = 0\ \forall s \ge 0\}$, then E remains invariant under $(\widetilde{T}(t))_{t\ge 0}$. As a consequence, we obtain the following lemma.

Lemma 3.4 ([12], Lemma 2.4). *The semigroup $(T_0(t))_{t\geq 0}$ induced by $(\widetilde{T}(t))_{t\geq 0}$ on E is*

$$(T_0(t)f)(s) = \begin{cases} U(s,s+t)f(t+s), & s+t \leq 0, \\ 0, & s+t > 0, \end{cases}$$

for $f \in E$.

The following lemma characterizes the generator of this semigroup.

Lemma 3.5 ([12], Lemma 2.5). *The generator $(G_0, D(G_0))$ of $(T_0(t))_{t\geq 0}$ is given by*

$$D(G_0) = \{f \in D(\widetilde{G}) \cap E \,:\, f(0) = 0\}, \qquad G_0 f = Gf.$$

We thus end up with operators $(G_0, D(G_0)) \subset (G, D(G)) \subset (\widetilde{G}, D(\widetilde{G}))$, where only the first and the third are generators on E and $\widetilde{E}$, respectively.
Using the operators $(G, D(G))$ and $(B, D(B))$, we can define a new operator on the product space $\mathcal{E}$.

Definition 3.6. On the product space $\mathcal{E} = X \times L^p(\mathbb{R}_-, X)$, define the operator $\mathcal{C}$ as

$$\mathcal{C} := \mathcal{C}_0 + \mathcal{F} := \begin{pmatrix} B & 0 \\ 0 & G \end{pmatrix} + \begin{pmatrix} 0 & \Phi \\ 0 & 0 \end{pmatrix}$$

with domain

$$D(\mathcal{C}) = D(\mathcal{C}_0) := \left\{ \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D(B) \times D(G) \,:\, f(0) = x \right\}$$

and $\mathcal{F} \in \mathcal{L}(D(\mathcal{C}_0), \mathcal{E})$.

Since $D(G) \subseteq C_0(\mathbb{R}_-, X) \cap L^p(\mathbb{R}, X) \subseteq D(\Phi)$, then Φf is well-defined. Recall that the last inclusion holds by General assumptions 3.1.

Now we want to recall two important results from [12] that will be used in this paper.

Proposition 3.7 ([12], Proposition 4.2). *The operator $(\mathcal{C}_0, D(\mathcal{C}_0))$ generates a strongly continuous semigroup $(\mathcal{T}_0(t))_{t\geq 0}$ on $\mathcal{E}$ given by*

$$\mathcal{T}_0(t) := \begin{pmatrix} S(t) & 0 \\ S_t & T_0(t) \end{pmatrix},$$

where $(T_0(t))_{t\geq 0}$is as in Lemma 3.4, $(S(t))_{t\geq 0}$ is the semigroup generated by $(B, D(B))$ and $S_t : X \to E$ is defined as

$$(S_t x)(\tau) := \begin{cases} U(\tau, 0)S(t+\tau)x, & t+\tau > 0, \\ 0, & t+\tau \leq 0. \end{cases} \tag{3.2}$$

Theorem 3.8 ([12], Theorem 4.5). *Assume that the delay operator Φ satisfies the Miyadera-Voigt condition, i.e.*

$$\int_0^{t_0} \|\Phi(S_r x + T_0(r)f)\| dr \leq q \left\| \begin{pmatrix} x \\ f \end{pmatrix} \right\| \tag{M}$$

for all $\begin{pmatrix} x \\ f \end{pmatrix} \in D(\mathcal{C}_0)$ *and some* $t_0 > 0$ *and* $0 \le q < 1$.

Then the operator $(\mathcal{C}, D(\mathcal{C}))$ *generates a strongly continuous semigroup* $(\mathcal{T}(t))_{t\ge 0}$ *on* $\mathcal{E}$ *satisfying*

$$\mathcal{T}(t)\begin{pmatrix} x \\ f \end{pmatrix} = \mathcal{T}_0(t)\begin{pmatrix} x \\ f \end{pmatrix} + \int_0^t \mathcal{T}(t-s)\mathcal{F}\mathcal{T}_0(s)\begin{pmatrix} x \\ f \end{pmatrix} ds$$

for all $\begin{pmatrix} x \\ f \end{pmatrix} \in D(\mathcal{C}_0), t \ge 0$, *with* $(\mathcal{T}_0(t))_{t\ge 0}$ *as in Proposition 3.7.*

From now on we always make the following additional assumption.

Assumption 3.9. The delay operator Φ satisfies condition (M).

The proof of the following proposition is an immediate consequence of Theorem 3.8 and of the definitions of $(T_0(t))_{t\ge 0}$ and of the function $t \mapsto S_t$ (see Lemma 3.4 and (3.2)).

Proposition 3.10. *The projections of* $(\mathcal{T}(t))_{t\ge 0}$ *onto the first and the second component on* $\mathcal{E}$ *satisfy the following identities*

$$\pi_1(\mathcal{T}(t)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)) = e^{tB}x + \int_0^t e^{(t-\tau)B}\Phi\pi_2(\mathcal{T}(\tau)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right))\, d\tau, \tag{3.3}$$

$$\pi_2(\mathcal{T}(t)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right))(s) = U(s, s+t)f(s+t) \tag{3.4}$$

if $s + t \le 0$, *and*

$$\pi_2(\mathcal{T}(t)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right))(s) = U(s,0)e^{(s+t)B}x + \int_0^{s+t} U(s,0)e^{(s+t-\tau)B}\Phi\pi_2(\mathcal{T}(\tau)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right))\, d\tau \tag{3.5}$$

if $s + t \ge 0$ *for all* $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D(\mathcal{C})$.

Remark 3.11.

1. Observe that $\pi_2(\mathcal{T}(t)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)) \in D(\Phi)$ since $\pi_2(\mathcal{T}(t)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)) = \widetilde{u}_t$ a.e. (see Corollary 3.14) and $\widetilde{u}_t \in D(\Phi)$ (see Remark 3.13).
2. It is interesting to compare these relations and the equality satisfied by the semigroup $(T_{B,\Phi}(t))_{t\ge 0}$ introduced by S. Brendle and R. Nagel. In particular they prove the following proposition.

Proposition 3.12 ([6], Proposition 2.14). *The semigroup* $(T_{B,\Phi}(t))_{t\ge 0}$ *satisfies*

$$(T_{B,\Phi}(t)f)(s) = U(s, s+t)f(s+t)$$

if $s + t \le 0$, *and*

$$(T_{B,\Phi}(t)f)(s) = U(s,0)e^{(s+t)B}f(0) + \int_0^{s+t} U(s,0)e^{(s+t-\tau)B}\Phi T_{B,\Phi}(\tau)f\, d\tau$$

if $s + t \ge 0$.

Remark 3.13. In [12, Theorem 3.5] it is proved that if $u(t)$ is given by

$$u(t) := \begin{cases} \pi_1\left(\mathcal{T}(t)\binom{x}{f}\right), & t \geq 0, \\ f(t), & \text{a.e. } t \leq 0, \end{cases} \tag{3.6}$$

then u is a **classical solution** of (NDE) for every $\binom{x}{f} \in D(\mathcal{C})$, i.e. the function $u : \mathbb{R} \to X$ satisfies the following properties:

(i) $u \in C(\mathbb{R}, X) \cap C^1(\mathbb{R}_+, X)$,

(ii) $u(t) \in D(B)$, $\tilde{u}_t \in D(\Phi)$, $t \geq 0$,

(iii) u satisfies (NDE) for all $t \geq 0$.

If $\binom{x}{f} \in \mathcal{E}$, then u can be considered as a **mild solution** of (NDE).

The reason for this last terminology is the following corollary.

Corollary 3.14. *The function $u : \mathbb{R} \to X$ defined in (3.6) for every $\binom{x}{f} \in \mathcal{E}$ satisfies the integral equation*

$$u(t) = \begin{cases} x + B\int_0^t u(s)ds + \Phi\int_0^t \tilde{u}_s ds, & t \geq 0, \\ f(t), & a.e.\ t \in \mathbb{R}_-, \end{cases}$$

where $\widetilde{u}_s$ is as in Introduction.

Proof. Let π_2 be the projection onto the second component of $\mathcal{E}$, i.e., $\pi_2\binom{x}{f} := f$ for all $\binom{x}{f} \in \mathcal{E}$.
First Step. We prove that

$$\widetilde{u}_t = \pi_2(\mathcal{T}(t)\tbinom{x}{f}) \quad \text{a.e.}. \tag{3.7}$$

Indeed, (3.7) holds by [12, Proposition 3.6] for $\binom{x_n}{f_n} \in D(\mathcal{C})$. Take now $\binom{x}{f} \in \mathcal{E}$ and a sequence $\binom{x_n}{f_n} \in D(\mathcal{C})$ converging to $\binom{x}{f}$. Since the semigroup $(\mathcal{T}(t))_{t\geq 0}$ is strongly continuous, the sequence $\mathcal{T}(t)\binom{x_n}{f_n}$ converges to $\mathcal{T}(t)\binom{x}{f}$ in $\mathcal{E}$.

Let

$$u_n(t) := \begin{cases} \pi_1(\mathcal{T}(t)\binom{x_n}{f_n}), & t \geq 0, \\ f_n(t), & \text{a.e. } t \leq 0. \end{cases}$$

Since $\binom{x_n}{f_n} \in D(\mathcal{C})$, we have $(\widetilde{u}_n)_t = \pi_2(\mathcal{T}(t)\binom{x_n}{f_n})$.

Moreover, if $-t \leq s \leq 0$,

$$(\widetilde{u}_n)_t(s) = \widetilde{U}(s, s+t)u_n(s+t) = \widetilde{U}(s, s+t)\pi_1(\mathcal{T}(t+s)\tbinom{x_n}{f_n}).$$

By our assumptions, it follows that $\|(\widetilde{u}_n)_t - \widetilde{u}_t\|_p \to 0$ as $n \to +\infty$. Thus, there exists a subsequence $(\widetilde{u}_{n_k})_t$ of $(\widetilde{u}_n)_t$ such that $(\widetilde{u}_{n_k})_t(s) \to (\widetilde{u}_t)(s)$ a.e..

Since

$$(\widetilde{u}_{n_k})_t(s) = \pi_2(\mathcal{T}(t)\tbinom{x_{n_k}}{f_{n_k}})(s) \to \pi_2(\mathcal{T}(t)\tbinom{x}{f})(s),$$

we can conclude that
$$\widetilde{u}_t = \pi_2(\mathcal{T}(t)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)) \quad \text{a.e..}$$
If $s \le -t$, one has
$$(\widetilde{u}_n)_t(s) = \widetilde{U}(s, s+t)u_n(s+t) = \widetilde{U}(s, s+t)f_n(s+t).$$
Since $\|f_n - f\|_p \to 0$, there exists a subsequence f_{n_k} of f_n such that $f_{n_k}(s) \to f(s)$ a.e.. Thus
$$\begin{aligned}(\widetilde{u}_{n_k})_t(s) &= \widetilde{U}(s, s+t)u_{n_k}(s+t) = \widetilde{U}(s, s+t)f_{n_k}(s+t) \\ &\to \widetilde{U}(s, s+t)f(s+t) = (\widetilde{u}_t)(s) \quad \text{a.e. for } s \le -t.\end{aligned}$$
Proceeding as above, we have
$$\widetilde{u}_t = \pi_2(\mathcal{T}(t)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)) \quad \text{a.e..}$$

Second step. Taking the first component of the identity
$$\mathcal{T}(t)\begin{pmatrix} x \\ f \end{pmatrix} - \begin{pmatrix} x \\ f \end{pmatrix} = \mathcal{C}\int_0^t \mathcal{T}(s)\begin{pmatrix} x \\ f \end{pmatrix} ds,$$
one has
$$\begin{aligned} u(t) - x &= \pi_1\left[\mathcal{C}\begin{pmatrix} \int_0^t \pi_1(\mathcal{T}(s)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right))ds \\ \int_0^t \pi_2(\mathcal{T}(s)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right))ds \end{pmatrix}\right] \\ &= \pi_1\left[\begin{pmatrix} B & \Phi \\ 0 & G \end{pmatrix}\begin{pmatrix} \int_0^t u(s)ds \\ \int_0^t \widetilde{u}_s ds \end{pmatrix}\right] = B\int_0^t u(s)ds + \Phi\int_0^t \widetilde{u}_s ds \end{aligned}$$
for all $t \ge 0$. ■

In the next section we discuss properties of the projection of $(\mathcal{T}(t))_{t\ge0}$ onto the second component of $\mathcal{E}$.

4 Classical Solutions for PDEs on $L^p(\mathbb{R}_-, X)$

In this section we want to exhibit classical solutions of (1.1) and (1.2) on $L^p(\mathbb{R}_-, X)$. As in [11], the basic idea is to find a core of $\mathcal{C}$, i.e., a dense set $\mathcal{D}$ in the domain $D(\mathcal{C})$, endowed with the graph norm, which is invariant under the semigroup $(\mathcal{T}(t))_{t\ge0}$. Since
$$D(\mathcal{C}) = \left\{\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D(B) \times D(G) \;:\; f(0) = x\right\}$$
(see Definition 3.6), the basic idea is to find a core of the operator G given in Definition 3.3. To this purpose we put
$$D_0 := \{f \in W^{1,p}(\mathbb{R}_-, X) : f(0) = 0, f(s) \in Y_s, s \mapsto A(s)f(s) \in E\}. \tag{4.1}$$
As in [25, Proposition 1.13] one can prove that the set D_0 is a core of G_0 in $L^p(\mathbb{R}_-, X)$, where G_0 is as in Lemma 3.5. The next result allows us to obtain a core of G.

Let $(U(t,s))_{t\le s\le 0}$ be as in Introduction and $\omega_0(\mathcal{U})$ its growth bound.

Proposition 4.1. *For $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega_0(\mathcal{U})$, the set*

$$D := D_0 \oplus \{\epsilon_\lambda y : y \in D(B)\} \tag{4.2}$$

is a core of G. Moreover,

$$Gf = f' + A(\cdot)f \quad a.e.$$

for every $f \in D$.

The proof of this proposition is based on the following two lemmas.

Lemma 4.2 (see [12], Lemma 4.1). *Let $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega_0(\mathcal{U})$. Then $\epsilon_\lambda x$ is an eigenvector of G with eigenvalue λ for every $x \in X$.*

Lemma 4.3. *Let $u \in E$ and $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega_0(\mathcal{U})$. Then $u \in D(G)$ and $\lambda u - Gu = 0$ if and only if $u(t) = (\epsilon_\lambda u(0))(t)$, $t \leq 0$.*

The previous lemma can be proved as in [17, Lemma 2.5].

Proof of Proposition 4.1. We first prove that D is dense in $D(G)$ with respect to the graph-norm, i.e.,

$$\forall\, f \in D(G) \text{ and } \forall\, \epsilon > 0 \text{ there exists } g \in D \text{ such that } \|f - g\|_G < \epsilon.$$

Let λ be such that $\Re\lambda > \omega_0(T_0(\cdot))$, hence $\lambda \in \rho(G_0)$. Since $G_0 = G_{|\mathrm{Ker}L}$, where $L : D(G) \to X$ is given by $Lf := f(0)$, we can apply a result proved by G. Greiner (see [16, Lemma 1.2]), obtaining that

$$D(G) = D(G_0) \oplus \mathrm{Ker}(\lambda - G).$$

Thus, using Lemma 4.2 and Lemma 4.3, for every $f \in D(G)$ there exists $f_0 \in D(G_0)$ and $x \in X$ such that $f = f_0 + \mu\epsilon_\lambda x$, for some constant μ.

Since ϵ_λ is bounded, there exists $M \in \mathbb{R}_+$ such that $\|\epsilon_\lambda\|_{\mathcal{L}(X,E)} \leq M$. Let $k_\lambda := 1 + M(1 + |\lambda|)$ and $\epsilon' := \frac{\epsilon}{k_\lambda}$.

Since $D(B)$ and D_0 are dense in X and $D(G_0)$, respectively, there exist $x_0 \in D(B)$ and $g_0 \in D_0$ such that

$$\|x - x_0\|_X < \epsilon'$$

and

$$\|f_0 - g_0\|_{G_0} < \epsilon'.$$

Let $g := g_0 + \mu\epsilon_\lambda x_0$. Then $g \in D$ and

$$\begin{aligned}
\|f - g\|_G &= \|f_0 - g_0\|_G + \|\epsilon_\lambda x - \epsilon_\lambda x_0\|_G = \|f_0 - g_0\|_{G_0} + \|\epsilon_\lambda x - \epsilon_\lambda x_0\|_G \\
&\leq \epsilon' + \|\epsilon_\lambda x - \epsilon_\lambda x_0\|_E + \|G\epsilon_\lambda x - G\epsilon_\lambda x_0\|_E \\
&\leq \epsilon' + \|\epsilon_\lambda\|_{\mathcal{L}(X,E)}\|x - x_0\|_X + |\lambda|\|\epsilon_\lambda\|_{\mathcal{L}(X,E)}\|x - x_0\|_X \\
&\leq \epsilon' + M(1 + |\lambda|)\epsilon' = k_\lambda\epsilon' = \epsilon.
\end{aligned}$$

Moreover, $Gf = f' + A(\cdot)f$ for every $f \in D$. In fact, if $f \in D$, write $f = f_0 + \mu\epsilon_\lambda x_0$ where $f_0 \in D_0$ and $x_0 \in D(B)$.

Then

$$Gf = G(f_0 + \mu\epsilon_\lambda x_0) = f_0' + A(\cdot)f_0 + \mu G\epsilon_\lambda x_0 = f_0' + A(\cdot)f_0 + \mu\lambda\epsilon_\lambda x_0$$

and

$$f' + A(\cdot)f = f_0' + (\mu\epsilon_\lambda x_0)' + A(\cdot)f_0 + \mu A(\cdot)\epsilon_\lambda x_0.$$

Since

$$(\mu\epsilon_\lambda x_0)'(s) = \mu\lambda(\epsilon_\lambda x_0)(s) + \mu e^{\lambda s}\frac{\partial}{\partial s}U(s,0)x_0 = \mu\lambda(\epsilon_\lambda x_0)(s) - \mu A(s)(\epsilon_\lambda x_0)(s),$$

it follows that $Gf = f' + A(\cdot)f$ for every $f \in D$. ■

The following lemma gives another expression for D.

Lemma 4.4. *The core D of $D(G)$, defined in (4.2), coincides with*

$$C := \{f \in W^{1,p}(\mathbb{R}_-, X) : f(0) \in D(B), f(s) \in Y_s, s \mapsto A(s)f(s) \in L^p(\mathbb{R}_-, X)\}.$$

Proof. "$\supseteq$" Let $f \in C$ and put

$$g := f - \epsilon_\lambda f(0).$$

Using General Assumptions 3.1.3 on the operators $A(s)$, it is easy to prove that $g \in D_0$. In fact, $g(0) = 0$, $g \in W^{1,p}(\mathbb{R}_-, X)$ and the function $s \mapsto A(s)g(s) = A(s)f(s) + A(s)(\epsilon_\lambda f(0))(s) \in L^p(\mathbb{R}_-, X)$. Since $U(s,0)Y_0 \subseteq Y_s$, then $g(s) \in Y_s$. Thus $f = g + \epsilon_\lambda f(0) \in D$.

"$\subseteq$" Let $f \in D$. Then $\exists\ f_0 \in D_0$ and $x \in D(B)$ such that $f = f_0 + \mu\epsilon_\lambda x$.

One has $f(0) = \mu x \in D(B)$, and since $U(s,0)Y_0 \subseteq Y_s$, it follows that $f(s) \in Y_s$. Moreover, by the General Assumptions 3.1.3 on the operators $A(s)$, we have that the function $s \mapsto A(s)f(s)$ belongs to $L^p(\mathbb{R}_-, X)$. ■

We are now ready to answer the problem posed at the end of the previous section. We have seen that the projection of $(\mathcal{T}(t))_{t\geq 0}$ onto the first component of $\mathcal{E}$ can be considered as a classical or a mild solutions of (NDE) (see Remark 3.13). The projection of $(\mathcal{T}(t))_{t\geq 0}$ onto the second component of $\mathcal{E}$ gives, instead, classical solutions of (1.1) and (1.2), i.e. $v(t,s) := \pi_2(\mathcal{T}(t)\binom{x}{f})(s)$, for appropriate $\binom{x}{f}$, belongs to the domain $D(G)$ of the operator G, is continuously differentiable, and satisfies (1.1) and (1.2).

To prove this result we consider

$$\mathcal{D} := \{\tbinom{x}{f} \in D(B) \times D : f(0) = x\}$$

as a subspace of $D(\mathcal{C})$. With a technique similar to the one used in [11] we can prove the following lemma.

Lemma 4.5. *Take $\Phi \in \mathcal{L}(C_0(\mathbb{R}_-, X)\cap L^p(\mathbb{R}_-, X), X)$, where $C_0(\mathbb{R}_-, X)\cap L^p(\mathbb{R}_-, X)$ is endowed with the norm $\|\cdot\|_\infty + \|\cdot\|_p$, and assume that Φ is bounded with respect to the L^p-norm, $1 \le p < +\infty$. If the functions*

$$s \mapsto A(s)U(s, s+t)f(s+t)$$

and

$$s \mapsto A(s)U(s, 0)g(s)$$

belong to $L^p([-t, 0], X)$ for all $f(\cdot) \in L^p(\mathbb{R}_-, X)$ and $g(\cdot) \in C([-t, 0], D(B))$, then the space $\mathcal{D}$ defined above is a $\mathcal{T}-$invariant subspace of $D(\mathcal{C})$.

Proof. Let $\binom{x}{f} \in \mathcal{D}$, then $\mathcal{T}(t)\binom{x}{f} \in D(\mathcal{C})$. Thus $\pi_1(\mathcal{T}(t)\binom{x}{f}) \in D(B)$, $\pi_2(\mathcal{T}(t)\binom{x}{f}) \in D(G)$ and $\pi_2(\mathcal{T}(t)\binom{x}{f})(0) = \pi_1(\mathcal{T}(t)\binom{x}{f})$. It remains to prove that $\pi_2(\mathcal{T}(t)\binom{x}{f}) \in D$. To this aim, we consider two cases.

First case. For $s \ge -t$, by Proposition 3.10, we can write $\pi_2(\mathcal{T}(t)\binom{x}{f})(s)$ as

$$\pi_2(\mathcal{T}(t)\tbinom{x}{f})(s) = U(s, 0)g_t(s),$$

where $g_t(s) := e^{(s+t)B}x + \int_0^{s+t} e^{(s+t-\tau)B}\Phi\pi_2(\mathcal{T}(\tau)\binom{x}{f})\, d\tau$. Since $\binom{x}{f} \in D(\mathcal{C})$, then $\pi_2(\mathcal{T}(t)\binom{x}{f})(0) \in D(B)$ and the function

$$\mathbb{R}_+ \ni \tau \mapsto \Phi\pi_2(\mathcal{T}(\tau)\tbinom{x}{f}) \in X \tag{4.3}$$

is continuous.

It follows that $g_t(\cdot) \in C([-t, 0], D(B))$.

Hence $g_t(s) \in Y_0$ for $s \in [-t, 0]$, since $D(B) \subseteq Y_0$. By assumption, we have that $U(s, 0)Y_0 \subseteq Y_s$, so $\pi_2(\mathcal{T}(t)\binom{x}{f})(s) \in Y_s$ and

$$\begin{aligned}\partial_s(\pi_2(\mathcal{T}(t)\tbinom{x}{f})(s) = &- A(s)U(s, 0)g_t(s) + U(s, 0)Bg_t(s) \\ &+ U(s, 0)\Phi\pi_2(\mathcal{T}(t+s)\tbinom{x}{f}).\end{aligned} \tag{4.4}$$

Hence the map $s \mapsto (\pi_2(\mathcal{T}(t)\binom{x}{f}))(s)$ is differentiable. In order to prove the thesis, it remains to show that the functions

1. $[-t, 0] \ni s \mapsto (\pi_2(\mathcal{T}(t)\binom{x}{f}))'(s)$,

2. $[-t, 0] \ni s \mapsto (A(\cdot)\pi_2(\mathcal{T}(t)\binom{x}{f})(s)$

are in $L^p(\mathbb{R}_-, X)$.

First, we prove (2).

It is obvious that $(A(\cdot)\pi_2(\mathcal{T}(t)\binom{x}{f}))(s) \in X$, since

$$(A(\cdot)\pi_2(\mathcal{T}(t)\tbinom{x}{f}))(s) = A(s)(\pi_2(\mathcal{T}(t)\tbinom{x}{f})(s)$$

and $(\pi_2(\mathcal{T}(t)\binom{x}{f})(s) \in Y_s \subseteq D(A(s)) \in X$. Since $(\pi_2(\mathcal{T}(t)\binom{x}{f})(s) = U(s, 0)g_t(s)$, it follows by the assumption in the theorem that $s \mapsto A(s)(\pi_2(\mathcal{T}(t)\binom{x}{f})(s)$ is in $L^p(\mathbb{R}_-, X)$.

Now, since $\pi_2(\mathcal{T}(t)\binom{x}{f})(0) \in D(B)$ and the function in 2. is in $L^p(\mathbb{R}_-, X)$, by (4.4) it is an immediate consequence that the function in 1. is also in $L^p(\mathbb{R}_-, X)$.

Second case. For $s < -t$, by Proposition 3.10, we can write $\pi_2(\mathcal{T}(t)\binom{x}{f})(s)$ as

$$\pi_2(\mathcal{T}(t)\tbinom{x}{f})(s) = U(s, s+t)f(s+t)$$

and obtain

$$\begin{aligned}\partial_s(\pi_2(\mathcal{T}(t)\tbinom{x}{f})(s) = &- A(s)U(s, s+t)f(t+s) + U(s, s+t)A(s+t)f(s+t) \\ &+ U(s, s+t)f'(s+t).\end{aligned} \tag{4.5}$$

As before, using the assumption on $A(s)$, we can show that the functions in 1. and 2. are in $L^p(\mathbb{R}_-, X)$ for $s < -t$.

Combining the two cases we conclude that $\mathcal{D}$ is $\mathcal{T}-$invariant. ■

Using the previous proposition, we can prove our main result as in [11, Theorem 4.4].

Theorem 4.6. *If the Assumption 3.9 and the assumptions of the previous lemma hold, then the function*

$$(t, s) \mapsto v(t, s) := \pi_2\left(\mathcal{T}(t)\tbinom{x}{f}\right)(s)$$

is the unique classical solution of (1.1) and (1.2) whenever $\binom{x}{f} \in \mathcal{D}$.

Here $(\mathcal{T}(t))_{t\geq 0}$ is the semigroup generated by the operator $(\mathcal{C}, D(\mathcal{C}))$ (see Definition 3.6 and Theorem 3.8).

Proof. Let $\mathbb{R}_+ \ni t \mapsto \mathcal{U}(t) := \binom{z(t)}{v(t)} \in \mathcal{E}$ a classical solution of the following Cauchy problem

$$(CP) \quad \begin{cases} \dot{\mathcal{U}}(t) &= \mathcal{C}\mathcal{U}(t), \quad t \geq 0, \\ \mathcal{U}(0) &= \binom{x}{f}, \end{cases}$$

for $\binom{x}{f} \in \mathcal{D} \subseteq D(\mathcal{C})$ (observe that the existence of this solution is guaranteed by Theorem 3.8).

In particular

$$\mathcal{U}(t) = \mathcal{T}(t)\tbinom{x}{f}, \tag{4.6}$$

for $t \geq 0$. Now consider the function

$$(t, s) \mapsto v(t, s) := \pi_2(\mathcal{T}(t)\tbinom{x}{f})(s), \tag{4.7}$$

i.e., using (4.7), $v(t, s) = v(t)(s)$, where $v(t)$ is the second component of $\mathcal{U}(t)$.

Since $\mathbb{R}_+ \ni t \mapsto \mathcal{U}(t) := \binom{z(t)}{v(t)} \in \mathcal{E}$ is a classical solution of (CP) with initial value $\binom{x}{f}$, we obtain that $v(t, s)$ is continuously differentiable with respect to t and s and

$$Gv(t, \cdot) = \frac{d}{dt}v(t, \cdot).$$

Moreover, since $\binom{x}{f} \in \mathcal{D}$, and using Lemma 4.5 and Proposition 4.1, one has

$$G(v(t,\cdot))(0) = Bv(t,0) + \Phi v(t,\cdot)$$

and

$$G(v(t,\cdot))(s) = \frac{\partial}{\partial s} v(t,s) + A(s)v(t,s).$$

So, $v(t,s)$ satisfies the two equations (1.1) and (1.2).

For the uniqueness, we assume that $v(\cdot,\cdot)$ is a solution of (1.1) and (1.2) for the initial value $v(0,\cdot) = f \in D$. From (1.2) and using the fact that $(U(t,s))_{t\leq s\leq 0}$ solves a nonautonomous Cauchy problem, we obtain

$$\begin{aligned} \frac{\partial}{\partial s} U(r,s)v(t,s) &= U(r,s)\left[A(s)v(t,s) + \frac{\partial}{\partial s} v(t,s)\right] \\ &= U(r,s)\frac{\partial}{\partial t} v(t,s) = \frac{\partial}{\partial t} U(r,s)v(t,s) \end{aligned} \tag{4.8}$$

for $r \leq s \leq 0$. Consequently, the expression

$$U(r,s)v(t,s)$$

can be written as a function of r and $s+t$. From this, it follows that

$$U(r,s)v(t,s) = \begin{cases} U(r,s+t)v(0,s+t), & s+t \leq 0, \\ U(r,0)v(s+t,0), & s+t \geq 0, \end{cases} \tag{4.9}$$

for $r \leq s \leq 0$. Putting $r = s$, we obtain

$$v(t,s) = \begin{cases} U(s,s+t)v(0,s+t), & s+t \leq 0, \\ U(s,0)v(s+t,0), & s+t \geq 0. \end{cases} \tag{4.10}$$

By equation (1.2) we have

$$\frac{d}{dt} v(t,0) = Bu(t,0) + \Phi v(t,\cdot).$$

Therefore, using the fact that $v(0,\cdot) = f$, we obtain

$$v(t,0) = e^{tB} f(0) + \int_0^t e^{(t-\tau)B} \Phi v(\tau,\cdot) d\tau. \tag{4.11}$$

Thus, by (4.9) and (4.10), we have

$$v(t,s) = \begin{cases} U(s,s+t)f(s+t), & s+t \leq 0, \\ U(s,0)e^{(t+s)B} f(0) + \int_0^{t+s} e^{(t+s-\tau)B} \Phi v(\tau,\cdot) d\tau, & s+t \geq 0. \end{cases} \tag{4.12}$$

Let now $f \equiv 0$. Using Gronwall's inequality (see [9, Lemma 2.A]), we see that $v(t,s) \equiv 0$. ∎

As an immediate consequence of [12, Theorem 3.5], one obtains the following corollary.

Corollary 4.7. *Let* $\mathbb{R}_+ \ni t \mapsto u(t)$ *a classical solution of* (NDE)*. If the Assumption 3.9 and the assumptions of Lemma 4.5 hold, then the function*

$$(t, s) \mapsto v(t, s) := \widetilde{u}_t(s)$$

is the unique classical solution of (1.1) and (1.2) whenever $\widetilde{u}_t \in D$*. Here* $\widetilde{u}_t$ *is the modified history function defined in Introduction.*

Example 4.8. Take $E := L^2(\mathbb{R}_-, X)$ as the Hilbert space X. Take $A(t) \equiv a(t)B$, where $a(\cdot) \in C(\mathbb{R}_-)$ with $a(t) > 0$ and $(B, D(B))$ a normal operator on X such that $s_0(B) < 0$. In this case, the evolution family is given by $U(t, s) = e^{(\int_t^s a(\tau)d\tau)B}$ and the regularity subspaces Y_t coincide with $D(B)$ for all $t \le 0$ (see [11, Example 4]). Let $1 < p < \infty$ and let $\eta : \mathbb{R}_- \to \mathcal{L}(X)$ be of bounded variation such that $|\eta|(\mathbb{R}_-) < +\infty$, where $|\eta|$ is the positive Borel measure in $\mathbb{R}_-$ defined by the total variation on η. Let $\Phi : C_0(\mathbb{R}_-, X) \cap L^p(\mathbb{R}_-, X) \to X$ be the linear operator given by the Riemann-Stieltjes integral

$$\Phi f := \int_{-\infty}^{0} f d\eta \quad \text{for all } f \in C_0(\mathbb{R}_-, X) \cap L^p(\mathbb{R}_-, X). \tag{4.13}$$

By [1, Proposition 1.9.4], this integral is well-defined. As in [12, Example 4.6] we can show that Φ fulfills the Miyadera-Voigt condition, i.e.,

$$\int_0^{t_0} \|\Phi(S_r x + T_0(r)f)\| dr \le q \left\| \begin{pmatrix} x \\ f \end{pmatrix} \right\|$$

for all $\begin{pmatrix} x \\ f \end{pmatrix} \in D(\mathcal{C}_0)$ and some $0 < t_0$ and $0 \le q < 1$.

By Theorem 3.8 the operator $(\mathcal{C}, D(\mathcal{C}))$ is the generator of a strongly continuous semigroup $(\mathcal{T}(t))_{t\ge0}$ and by Theorem 4.6 the function $(t, s) \mapsto v(t, s) := \pi_2\left(\mathcal{T}(t)\binom{x}{f}\right)(s)$ is a classical solutions of (1.1) and (1.2) for $\binom{x}{f} \in \mathcal{D} := \{\binom{x}{f} \in D(B) \times D : f(0) = x\}$, with $D := \{f \in W^{1,p}(\mathbb{R}_-, X) : f(s) \in D(B)\ \forall\ s \le 0, s \mapsto a(s)Bf(s) \in L^p(\mathbb{R}_-, X)\}$.

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