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The averaged moduli of smoothness with applications in numerical methods and approximation, by Blagovest Sendov and Vasil A. Popov. John Wiley & Sons, New York, 1988, 180 pp. ISBN 0-471-91952-7

For functions f, bounded (and Lebesgue measurable) on a compact interval $[a, b]$ of the real axis, the classical modulus of continuity (smoothness) may be introduced via

$$\omega_k(f, \delta)_\infty := \sup_{a\leq x\leq b} \omega_k(f, x, \delta), \tag{1}$$

using the kth local modulus of continuity

$$\omega_k(f, x, \delta) := \sup\left\{|\Delta_h^k f(t)| : t, t+kh \in [a, b] \cap \left[x - \frac{k\delta}{2}, x + \frac{k\delta}{2}\right]\right\},$$

$$\Delta_h^k f(t) := \sum_{j=0}^{k}(-1)^{k+j}\binom{k}{j} f(t+jh).$$

It is well known that (1) as well as its L^p-analogue $(1 \leq p < \infty)$

$$\omega_k(f, \delta)_p := \sup_{0\leq h\leq\delta}\left[\int_a^{b-kh} |\Delta_h^k f(x)|^p \, dx\right]^{1/p} \tag{2}$$

serve as a measure of smoothness of functions in many fields of analysis. In particular, the moduli (1; 2) often supply appropriate bounds for the error of approximation processes, given via sequences of bounded (e.g., integral) operators on L^p. When dealing with approximation procedures of a discrete structure, however, an estimation of L^p-errors (e.g., for Bernstein polynomials of bounded functions) versus $\omega_k(f, \delta)_p$ is not possible since, roughly speaking, $\omega_k(f, \delta)_p$ represents a bounded (sublinear) functional on L^p, whereas point evaluation functionals cannot be bounded with regard to the L^p-metric, even when restricted to continuous functions.

In this connection the authors offer the averaged or τ-modulus of smoothness

$$\tau_k(f, \delta)_p := \left[\int_a^b (\omega_k(f, x, \delta))^p \, dx\right]^{1/p} \tag{3}$$

as an appropriate substitute. This τ-modulus was first introduced around 1967–69 in work of Bl. Sendov and P. P. Korovkin in connection with approximation in the so-called Hausdorff distance. First of all, the τ-modulus possesses properties which are indeed quite analogous to those of the classical ones (1; 2). The interconnections are described via the estimates (cf. Theorem 1.4 of the book in question)

$$\omega_k(f, \delta)_p \leq \tau_k(f, \delta)_p \leq (b-a)^{1/p} \omega_k(f, \delta)_\infty .$$

To mention one of the interesting new features of the τ-modulus, for a bounded (L-measurable) function f to be Riemann integrable, it is necessary and sufficient that (cf. Theorem 1.2)

$$\lim_{\delta \to 0+} \tau_1(f, \delta)_1 = 0, \tag{4}$$

which corresponds nicely to the familiar fact that the continuity of f on $[a, b]$ is equivalent to $\lim_{\delta \to 0+} \omega_1(f, \delta)_\infty = 0$. As one consequence, estimating errors versus τ-moduli often allows one to reduce the assumptions actually to those, natural in the formulation of the problem at hand, for example, to Riemann integrability in connection with quadrature procedures. In fact, an estimate of the type (cf. Theorem 3.2)

$$|R_n^{Tr} f| \leq \tau_2(f, (b-a)/n)_1$$

for the error of the compound trapezoidal rule provides a quantitative description of the well-known fact that one has convergence for each Riemann integrable function (cf. (4)).

The book under review aims to propagate the usefulness of the new concept. Whereas the first two chapters present the definitions and the basic properties of τ-moduli, the following six chapters are devoted to applications in various fields of approximation theory and numerical analysis. Topics included are concerned with numerical integration, with the approximation in L^p of functions by means of linear summation operators of type (e.g., Bernstein polynomials, Lagrange interpolation, splines)

$$L_n(f; x) := \sum_{j=0}^{n} f(x_{jn}) \varphi_{jn}(x),$$

with the estimation of the error in the numerical solution of integral equations as well as of initial and boundary value problems for ordinary differential equations, and with a constructive theory of functions in the frame of one-sided approximation.

The first Bulgarian version of the book was published by the Publishing House of the Bulgarian Academy of Sciences, Sofia, in 1983. The present one is the carefully edited English translation together with several improvements. Now each chapter ends with a section on Notes, which not only provides some historical background to the subject, but also updates the material. Indeed, the list of references is considerably enlarged, and Section 2.1 on Whitney's theorem is completely rewritten, including the recent achievements of the senior author concerning the boundedness of Whitney's constants.

This book for the first time introduces into a field of recent progress in approximation theory and numerical analysis. Therefore it certainly will be of great value to those working in the broad area of error analysis. The book is well organized and (almost) self-contained. In fact, the presentation of the material is introductory, proofs are worked out in detail, and the pace is leisurely. Particularly in the applications, the authors do not present the most general results but try to emphasize the underlying principles in connection with significant examples. A list of symbols and an index round out this useful publication. In all, the book nicely surveys a substantial portion of the work of the very active Bulgarian school of approximation.

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Matrix norms and their applications, by G. R. Belitskii and Yu. I. Lyubich. Translated by A. Iacob. Oper. Theory: Adv. Appl., vol. 36, Birkhäuser, Basel, Boston, Berlin, 1988, viii+209 pp., $87.50. ISBN 0-8176-2220-9

The study of ring norms may be considered to go back to the well-known papers by Murray and von Neumann on rings of operators [4], by Gelfand on commutative Banach algebras [1], and by Gelfand and Naimark on C^*-algebras [2]. In a paper on "the metrization of matric-space" [5], von Neumann investigated the properties of ring norms constructed from gauge functions on $\mathbf{R}^n$;