The monograph gives a comprehensive exposition of this beautiful theory. Most of the results are in book form for the first time. It is well written and clearly presented. The author has tried to make it as self-contained as possible by including some introduction to the dilation theory and functional model of contractions. Each chapter starts with a lucid summary of what is to come and each section ends with a set of well-chosen exercises. It can serve as a graduate textbook after a standard functional analysis course, a book for seminar topics or a monograph for research reference. It is also a stepping stone toward a better understanding of Sz.-Nagy and Foiaş' contraction theory as presented in [**5**].

The reviewer noticed relatively few misprints. Some discrepancies of the terminology do occur: antilinear map (p. 37) is the same as conjugate linear map (p. 64); $\{T\}''$ (p. 74) has been called double commutant (p. 182) and bicommutant (p. 227). To the references he would suggest to add [**1**, **3**, and **6**].

In summary, the author has done an outstanding job presenting a part of operator theory which, because of its intrinsic interest and potential applications to systems theory, deserves more attention among practitioners in this field.

REFERENCES

1. D. N. Clark, *One dimensional perturbations of restricted shifts*, J. D'analyse Math. **25** (1972), 169–191.

2. K. R. Davidson and D. A. Herrero, *The Jordan form of a bitriangular operator*, preprint, 1988.

3. L. J. Gray, *Jordan representation for a class of nilpotent operators*, Indiana Univ. Math. J. **26** (1977), 57–64.

4. B. Sz.-Nagy and C. Foiaş, *Sur les contractions de l'espace de Hilbert.* VII. *triangulations canoniques, fonctions minimum*, Acta Sci. Math. (Szeged) **25** (1964), 12–37.

5. ——, *Harmonic analysis of operators on Hilbert space*, North-Holland, Amsterdam, 1970.

6. B. Sz.-Nagy, *Unitary dilations of Hilbert space operators and related topics*, CBMS Regional Conf. Ser. in Math., no. 19, Amer. Math. Soc., Providence, R.I., 1974.

7. L. R. Williams, *A quasisimilarity model for algebraic operators*, Acta Sci. Math. (Szeged) **40** (1978), 185–188.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 21, Number 1, July 1989

0273-0979/89 $1.00 + $.25 per page

Stratified Morse theory, by M. Goresky and R. MacPherson. Springer-Verlag, Berlin, Heidelberg, New York, 1988, xiv + 272 pp., $75.00. ISBN 3-540-17300-5

An important tool in the investigation of the topology of a differentiable manifold M is the classical Morse theory. Given a Morse function φ, i.e., a proper differentiable function $\varphi: M \to \mathbf{R}$, bounded from below and with isolated nondegenerate critical points, one considers the sets $M_{\leq a} := \{x \in M; \varphi(x) \leq a\}$ and asks how the topological type of $M_{\leq a}$ changes when a varies. One may always assume that the restriction of the mapping φ to its critical set is injective. If $\varphi^{-1}([a, b])$ does not contain a critical point, then $M_{\leq b}$ is diffeomorphic to $M_{\leq a}$; if $\varphi^{-1}([a, b])$ contains exactly one critical point p and if $a < \varphi(p) < b$, then $M_{\leq b}$ is homeomorphic to $M_{\leq a}$ with a handlebody $(D^\lambda, \partial D^\lambda) \times D^{n-\lambda}$ attached to it along $\partial D^\lambda \times D^{n-\lambda}$. Here n is the (real) dimension of the manifold M, and λ, the Morse index of φ at the point p, is the number of negative eigenvalues of the Hessian of φ at p.

In particular, one obtains information about the homotopy type of the manifold M, estimates for its Betti numbers, etc. If the differentiable manifold M is a complex manifold, then there is often a natural choice for such a Morse function, which has Morse indices less than or equal to the complex dimension of M. That fact leads to some remarkable consequences.

Unfortunately, in algebraic geometry and in complex analysis the varieties or complex spaces that have to be considered are not smooth in general! In this situation an expedient is offered by the observation that every singular variety Z may be stratified, i.e. written as a disjoint union $Z = \bigcup_i S_i$ of locally closed smooth subvarieties S_i. The idea is to apply classical Morse theory to each stratum S_i separately, but afterwards, of course, one has to tackle a rather delicate question, namely how to patch together the theory of the different pieces.

It is the achievement of Mark Goresky and Robert MacPherson that they were able to handle this problem with the creation of their stratified Morse theory. The book at hand presents that theory for the first time in a systematic manner. It is divided into three parts: the first section deals with the general theory, the second part is devoted to the complex case and carries over many central results of classical Morse theory from the smooth to the singular case, in the third part the homology of the complement of a finite system of affine subspaces in $\mathbf{R}^n$ is computed.

Let us start with the general theory. Morse theory will be developed on Whitney-stratified closed subsets $Z = \bigcup_i S_i \subset M$ of an ambient differentiable manifold M. A Morse function f on Z then is the restriction of a smooth real valued function defined near Z which is proper and bounded from below such that

(1) the restriction to every stratum has only isolated nondegenerate critical points

(2) at every critical point $p \in S_i$ the function f satisfies a regularity condition normal to S_i: the restriction of the differential $df(p)$ to every limit tangent plane of a higher dimensional stratum is nontrivial.

For a critical point p and the critical value $c = \varphi(p)$ Morse data are defined to be a pair of topological spaces (A, B) such that $M_{\leq c+\varepsilon}$ is obtained up to homeomorphy from $M_{\leq c-\varepsilon}$ by attaching A along B. The central result of the first part is the following description of the Morse data (A, B): they can be defined locally and there they admit a decomposition as a product of a tangential part arising from the stratum S containing p—a handlebody as in the classical situation—and a normal part associated to a normal slice N of S at p. The arguments in the proof of that statement involve a lot of demanding geometry; an essential idea is to use the technique of "moving the wall," which is based on Thom's first isotopy lemma and allows to compare the various types of Morse data that come up in the proof.—The stratification approach in Morse theory suggests naturally two generalizations, which even in the smooth case lead to new results: nonproper Morse functions $\varphi\colon Z \to \mathbf{R}$ can be treated in an analogous manner provided that $\overline{Z}$ admits a Whitney stratification such that Z is a union of strata and there exists a Morse function $\overline{\varphi}\colon \overline{Z} \to \mathbf{R}$ extending φ; on the other hand the theory applies also to relative situations, where the topology of a space X lying over Z by means of a proper stratified submersion $\pi\colon X \to Z$ is investigated. Finally, a combination of these two cases is considered as well.

In the second section the reader encounters a lot of beautiful applications which really reward him for his endurance. It is dedicated to the aspects of the theory in the complex setting. A special feature here is that normal Morse data for a critical point p depend only on the point p resp. on the connected component of the stratum containing p and no longer on the Morse function itself. In order to give a flavour of the results we mention the following points: the homotopy dimension of varieties can be estimated even in the relative and the nonproper case; in particular, a conjecture of Deligne is proved, Lefschetz theorems of the hyperplane type are treated, both in the local and the global situation; various consequences for intersection homology are discussed.

Eventually, in the third section we find an application of nonproper Morse functions: the homology of the complement of a finite system of affine subspaces in $\mathbf{R}^n$ is expressed in terms of simplicial (relative) cohomology of the associated order complex.

Apart from the fundamental importance of the new theory as an extremely powerful generalization of classical Morse theory, the book catches the reader by its presentation of the material. The authors discuss carefully the history of both Morse and stratification theory and give a precise account of what exists in the literature so far. Chapters and sections are well organized, many introductory remarks motivate definitions and explain the strategy of proofs; a lot of pictures and diagrams help the reader to understand the geometric ideas which stand behind them.

Summing up, the book will certainly very soon prove an indispensable reference for everybody working in this field, and it seems not unjustified to expect that the book might quickly become a classic of modern Morse theory.

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