There is a great deal more covered in Lang's book than I have discussed here, e.g., Nevanlinna theory and the defect relations of Carlson-Griffiths, surely the foremost success of differential-geometric methods in the subject. Indeed, there is little of importance in the area that Lang set out to cover which he has not managed to include in either his book or survey article. If I had a student in this area, I would surely point to these two sources and say, "This is what you ought to learn." There are, as one might expect in a work of this scope, better places to learn some of the topics covered; however, nowhere else are even half of these topics all to be found together. Chapters 3 and 7, for example, contain material available nowhere else in book form. Lang has performed a tremendously important service to the subject.

BIBLIOGRAPHY

1. S. Kobayashi, *Hyperbolic manifolds and holomorphic mappings*, Marcel Dekker, New York, 1970.

2. S. Lang, *Hyperbolic and diophantine analysis*, Bull. Amer. Math. Soc. (NS) **14** (1986), 159–205.

3. P. Vojta, *Diophantine approximations and value distribution theory*, Lecture Notes in Math., vol. 1239, Springer-Verlag, Berlin and New York, 1987.

MARK GREEN

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 18, Number 2, April 1988

0273-0979/88 $1.00 + $.25 per page

Spectral geometry: Direct and inverse problems, by Pierre H. Bérard, with an appendix by G. Besson. Lecture Notes in Mathematics, Vol. 1207, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, 1986, xiii + 272 pp., $23.40. ISBN 3-540-16788-9

This interesting book deals with certain direct problems in Riemannian geometry. The notions of direct and inverse problems are not exact, but in general, by a direct problem one means a problem in which the goal is to derive information about the spectrum of the Laplace operator on a Riemannian manifold M from other geometric data associated with the manifold, while an inverse problem is one in which the deduction goes the other way, i.e., information about the spectrum is used to derive different geometric information about M.

For example, the Faber-Krahn result that among smooth domains in R^n of fixed volume the first Dirichlet eigenvalue is minimized by the ball is of a direct character, whereas the fact that the dimension of such a domain is determined by its Dirichlet spectrum is of an inverse character.

There is a very large body of results on these topics, with certain persistent and central themes, e.g., the effort to understand the asymptotic behavior of the spectrum. Strikingly, this behavior only depends, up to an asymptotically correct first approximation, on the dimension and Riemannian volume of M (cf. [**M-P**]). A related development has been the discovery of relationships which, formally at least, convey the exact geometric content of the spectrum, or looked at the other way, display the complete spectrum in terms of other geometric quantities (cf. [**CDV**, **D**, **S**]). For example, it follows from the Selberg trace formula that within the class of compact surfaces of constant curvature -1, knowledge of the spectrum is exactly the same as knowledge of the volume and the length spectrum of M, the latter being the list, with correct multiplicities, of lengths of closed geodesics on M. The degree to which the information in these lists is redundantly coded is an interesting general question in its own right. For example, it is easy to show that the length spectrum determines the volume, and there are other redundancies as well (cf. [**MK**]). Another highly studied class of questions concerns the small, rather than large, eigenvalues of M. Such questions are, broadly speaking, of the isoperimetric type, and are exemplified, for example, by Cheeger's inequality, the Faber-Krahn inequality, and the Obata theorem. It is around questions of this latter sort, in the context of compact manifolds, that the themes of Bérard's book develop.

In more detail, the book is principally concerned with isoperimetric results obtained by the author, in collaboration with G. Besson and S. Gallot, and largely built around the exploitation of a particular technique of symmetrization. After three chapters of an introductory character, which convey a short but clear account of the parts of Riemannian geometry that will eventually be needed, Chapter 4 introduces the isoperimetric function of a manifold M: $h(\beta) = \inf_\Omega \operatorname{vol}(\partial\Omega)/\operatorname{vol}(\Omega)$, where $\beta \in (0,1)$, and Ω ranges over smooth domains in M for which $\operatorname{vol}(\Omega) = \beta \operatorname{vol}(M)$. Since it is very rare for $h(\beta)$ to be known explicitly, weaker information is presumed to be available, namely an estimator function $H(\beta)$ satisfying $h(\beta) \geq H(\beta)$. Assuming reasonable properties for $H(\beta)$, a new manifold M^* having rotational symmetry can then be constructed, in which the volume and area of balls about a distinguished point are related through $H(\beta)$. Now it follows from a theorem of the author and his collaborators, that for manifolds for which the diameter and the Ricci curvature are suitably related, the function $H(\beta)$ can be taken to be the isoperimetric function of a Euclidean sphere, so that for such manifolds, which we will call of class N, the ancillary manifold M^* becomes a Euclidean sphere. This makes practical the use of M^*, via various comparison techniques, for the estimation of quantities associated with manifolds in the class N. The most important application of this idea is presented in Chapter 5, where it is shown that for these manifolds, the theta-type function $\sum e^{-\lambda_j t}$, which is the trace of the heat kernel on M, can be dominated by the trace of the heat kernel of a suitable Euclidean sphere. Various lower bounds for the eigenvalues of manifolds of class N then follow as corollaries of this result. The next chapter (Chapter 6), takes up the consideration of generalizations to a vector bundle setting of the Bochner theorems relating the first Betti number and the Ricci curvature of a manifold. In particular, the above bound on the trace of the heat kernel is combined with other results to bound the dimension of the space of harmonic sections of appropriate vector bundles over manifolds of class N.

The book concludes with a short survey of recent developments, and an appendix by Besson detailing a formal approach via functional analysis to some of the above ideas, and finally presents the famous Guide to the Literature and Bibliography of Riemannian Geometry, compiled by Bérard and Berger, with a partial update covering the period since 1982.

The book is not, and is not intended to be, a broad overview of the by now very large topics of direct and inverse problems in Riemannian geometry. It is, however, a clear account of the contributions along the above lines of the author and his collaborators, and some of its material is not in print elsewhere. Altogether, within the framework of its aims, the book conveys a clear account of this interesting work, and comprises, together with the recent book of Chavel [CH] on related topics, a very worthwhile addition to the literature of spectral geometry.

BIBLIOGRAPHY

[CDV] Y. Colin de Verdière, *Spectre du Laplacien et longueurs des géodésiques périodiques.* I, II, Compositio Math. **27** (1973), 83–106; 159–184.

[CH] I. Chavel, *Eigenvalues in Riemannian geometry*, Academic Press, New York, 1984.

[D] J. Delsarte, *Sur le gitter Fuchsien*, C. R. Acad. Sci. Paris **241** (1942), 147–149.

[M-P] S. Minakshisundaram and øA. Pleijel, *Some properties of the eigenfunctions of the Laplace operator on Riemannian manifolds*, Canad. J. Math. **1** (1949), 242–256.

[MK] H. P. McKean, *Selberg's trace formula as applied to a compact Riemann surface*, Comm. Pure Appl. Math. **25** (1972), 225–246.

[S] A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric spaces with applications to Dirichlet series*, J. Indian Math. Soc. **20** (1956), 47–87.

BURTON RANDOL

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 18, Number 2, April 1988

0273-0979/88 $1.00 + $.25 per page

Intégration et théorie des nombres, by Jean-Loup Mauclaire; preface by S. Iyanagi. Hermann, Paris, 1986, xi + 152 pp., 160F. ISBN 2-7056-6035-6

A function is arithmetic if it is defined on the positive integers. In this review arithmetic functions will be real or complex valued. The scope of this definition is rather wide, and functions of number theoretic interest generally have some structure attached to them. An example is the Dirichlet divisor function $d(n)$, which counts the number of distinct divisors of the integer n. Its values on the first ten integers are 1, 2, 2, 3, 2, 4, 2, 4, 3, 4, and appear roughly increasing. Considered over the range $205 < n \leq 215$ however, we have 4, 6, 10, 4, 16, 2, 6, 4, 4, 4. It is characteristic of functions of number theoretic interest that their successive values sail so erratically about. I begin with a snapshot history of the methods devised in Analytic Number Theory to come to grips with this phenomenon. As in many a family album, some important relations do not get into the picture.

According to Dirichlet, it was Gauss who considered the mean-value

$$M(g,x) = x^{-1} \sum_{n \leq x} g(n) \tag{1}$$