of upper-lower solutions and monotone iterations. Periodic and terminal boundary conditions are included in the discussion. Chapter 3 is concerned with elliptic equations, and Chapter 4 with parabolic equations. A major part of these two chapters is devoted to the existence problem for systems where the nonlinear reaction function depends on u as well as on ∇u. These two chapters cover the main theme of the book and they deserve special attention. The final chapter treats the hyperbolic equation of first order. Here the method of upper-lower solutions is shown to be useful for the construction of a Lyapunov function. The book is self-contained, with an appendix giving most of the necessary material from the theory of linear partial differential equations. The bibliography is extensive, and it leads the reader to various references for more detailed discussions on related subjects.

Although there are some minor points which need more explanation or clarification, the book is well written and is a much needed and timely addition to the current literature, especially in the area of nonlinear reaction-diffusion systems. Despite the distinct characteristics among second-order elliptic, parabolic and hyperbolic equations, the authors have successfully established a unified approach and cast these problems into the same framework of monotone technique. This book may well stimulate further research in other areas of differential and integral equations and related fields. In fact, the monotone method and its associated upper and lower solutions have already been used for the treatment of numerical solutions of nonlinear parabolic and elliptic equations. It is likely that both the analytical techniques and the numerical schemes will receive even greater attention in various applied sciences.

C. V. PAO

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K-theory for operator algebras, by Bruce Blackadar, Mathematical Sciences Research Institute Publications, vol. 5, Springer-Verlag, New York, Berlin, Heidelberg, London, Paris, Tokyo, 1986, 338 pp., $28.00. ISBN 0-387-96391-x

The development of K-theory has been one of the great unifying forces in mathematics during the past thirty years, bringing together ideas from geometry, algebra, and operator theory in fruitful and often unexpected ways, and stimulating each of these subjects through the importation of insights and techniques from other areas.

It is commonly agreed that K-theory originated with the work of Grothendieck in the late 1950s in which he proved a generalized Riemann-Roch theorem. This involved the construction of a group from the category of coherent algebraic sheaves over a projective algebraic variety. Grothendieck's idea was taken up by Atiyah and Hirzebruch, who showed how to associate with any compact space X a group $K(X)$ constructed from the category of vector bundles over X. This soon became a vital tool in such areas as stable homotopy theory and index theory for elliptic operators. The space X is determined by the ring of continuous functions $C(X)$, and in this context the elements of $K(X)$ can be described in terms of the finitely-generated projective modules over $C(X)$. This result is due to Swan, although Serre had previously proved the analogous result for coherent sheaves over algebraic varieties. Thus for any unital ring R one can define $K(R)$ as the enveloping group of the semigroup of isomorphism classes of finitely-generated projective modules over R. Such a module is a direct summand of a free module, and by considering the projection from the free module to the projective module one arrives at the operator algebraist's viewpoint: for a unital C^*-algebra A, $K(A)$ is the enveloping group of the semigroup of equivalence classes of projections in matrix algebras over A. The semigroup operation is given by direct sums and the equivalence relation can be taken to be stable unitary equivalence ("stable" here means that one identifies a matrix algebra with its embedding in a larger matrix algebra).

The first explicit appearance of K-theory, in this form, in operator algebras did not occur until about ten years ago, following work of Elliott in which he used it to give a complete isomorphism invariant for AF algebras (inductive limits of finite-dimensional C^*-algebras). Elliott's result had earlier been proved for special classes of AF algebras by Glimm and Dixmier. But operator algebraists will recognize the basic elements in the construction of $K(A)$ as going back much further than that. The very first paper on rings of operators by Murray and von Neumann (1936) introduces the comparison theory of projections as the basis for classifying von Neumann algebras; and the technique of using matrix algebras over a given operator algebra goes back even further, to von Neumann's proof of the double commutant theorem (1929). With hindsight, it can seem quite surprising that von Neumann did not develop operator-algebraic K-theory fifty years ago.

Since $K(A)$ is constructed from a semigroup, it is not only a group but also carries an ordering, and one can ask which ordered groups arise in this way. In the case of AF algebras there is a very satisfactory answer to this question, due to Effros, Handelman, and Shen. They proved that an ordered group is a dimension group (that is, $K(A)$ for some AF algebra A) if and only if it is unperforated and satisfies the Riesz interpolation property. Using this, Bratteli, Elliott, and Herman were able to show that, given any closed subset K of $\mathbb{R} \cup \{\pm\infty\}$, there is a quantum dynamical system for which the set of inverse temperatures at which there exist equilibrium states is exactly K.

For a C^*-algebra A, define the *suspension* of A to be the C^*-algebra $SA = A \otimes C_0(\mathbb{R})$. We write $K_0(A) = K(A)$, $K_1(A) = K(SA)$ (having first extended the definition of $K(A)$ to cover the nonunital case). If

$$0 \to J \to A \to A/J \to 0$$

is an exact sequence of C^*-algebras, then there is a six-term exact sequence

$$\begin{array}{ccccc} K_0(J) & \longrightarrow & K_0(A) & \longrightarrow & K_0(A/J) \\ \uparrow & & & & \downarrow \\ K_1(A/J) & \longleftarrow & K_1(A) & \longleftarrow & K_1(J), \end{array}$$

in which the horizontal maps are defined functorially and the vertical connecting maps are very concretely specified (the index map and the exponential map). This is the C^*-algebraic formulation of Bott periodicity, and is the first deep result in the theory, providing a powerful tool for the computation of K-groups.

The next major developments require the notion of a crossed product for C^*-algebras. Suppose that G is a (locally compact, abelian) topological group which acts as a group of automorphisms of a C^*-algebra A in a suitably continuous way. Then the crossed product $A \rtimes G$ is a "larger" C^*-algebra in which these automorphisms "become inner". (This statement is correct if G is discrete; in the continuous case the phrases in quotes need to be interpreted a bit elastically.) For a very concrete example, let A be the commutative C^*-algebra $C(T)$ of continuous complex-valued functions on the circle, which acts (by pointwise multiplication) as an algebra of operators on the Hilbert space $L^2(T$, Haar measure). For a fixed irrational θ in $(0,1)$, let u be the unitary operator on $L^2(T)$ given by "rotation through θ":

$$ux(z) = x(ze^{-2\pi i\theta}) \qquad \left(x \in L^2(T), z \in \mathbf{C}, |z| = 1\right).$$

Then u induces an automorphism of A, and the group $\mathbf{Z}$ acts on A by powers of this automorphism. Define $A_\theta = A \rtimes \mathbf{Z}$. It can be shown that A_θ is a simple C^*-algebra, and that it is isomorphic to the C^*-algebra of operators on $L^2(T)$ generated by A and u. These "irrational rotation algebras" have been important in the development of the theory. Innocent little questions about them (Do they contain nontrivial projections? Are they isomorphic to each other, for different values of θ?) can be hard to answer, and have provoked major advances in C^*-algebraic K-theory.

There are two big theorems which provide information about K-theory for crossed products. The first, due to Pimsner and Voiculescu, deals with crossed products by $\mathbf{Z}$. Suppose that α is an automorphism of a C^*-algebra A, generating an action of $\mathbf{Z}$ on A. There is a six-term exact sequence

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{1-\alpha_*} & K_0(A) & \xrightarrow{\iota_*} & K_0(A \rtimes \mathbf{Z}) \\ \uparrow & & & & \downarrow \\ K_1(A \rtimes \mathbf{Z}) & \xleftarrow{\iota_*} & K_1(A) & \xleftarrow{1-\alpha_*} & K_1(A), \end{array}$$

where ι: $A \to A \rtimes \mathbf{Z}$ is the natural inclusion. This sequence does not completely determine the K-theory of $A \rtimes \mathbf{Z}$ in all cases, but it does provide a great deal of information. For example, it enables one to give good answers to the two questions in the previous paragraph about irrational rotation algebras.

The second big theorem is Connes' Thom Isomorphism Theorem, which says that for crossed products by $\mathbb{R}$,

$$K_i(A \rtimes \mathbb{R}) \cong K_{1-i}(A) \qquad (i = 0, 1).$$

This obviously provides complete information about the K-theory of $A \rtimes \mathbb{R}$, and in fact says that it is independent of the way $\mathbb{R}$ acts on A. In the special case when the action is trivial, the crossed product is just SA and we retrieve the Bott Periodicity Theorem.

We now turn to another main theme of operator algebraic K-theory, the study of extensions. In its most general setting, this seeks to classify short exact sequences

$$0 \to B \to E \to A \to 0,$$

where A and B are given C^*-algebras. In order to classify, one needs to know when to identify two extensions, and unfortunately there are many different choices for an equivalence relation on the set of extensions, all of them described in terms of the so-called Busby invariant, which we shall now define.

For a C^*-algebra B, there is a maximal C^*-algebra $M(B)$, the multiplier algebra of B, which contains B as an essential ideal. The quotient algebra $Q(B) = M(B)/B$ is called the outer multiplier algebra of B. Given an extension as in the previous paragraph, there is an evident homomorphism from E into $M(B)$. This, composed with the quotient maps, gives the Busby invariant τ: $A \to Q(B)$. Conversely, given τ, one can reconstruct the extension from the diagram

$$\begin{tikzcd} & & E \arrow[r] \arrow[d] & A \arrow[d, "\tau"] & \\ 0 \arrow[r] & B \arrow[r] & M(B) \arrow[r, "\pi"] & Q(B) \arrow[r] & 0 \end{tikzcd}$$

by taking E to be the pullback

$$E = \{(m, a) : m \in M(B), a \in A, \pi(m) = \tau(a)\}.$$

The simplest case to study is that in which B is the algebra K of compact operators on a separable Hilbert space H. Then $M(B)$ can be identified with $B(H)$ and $Q(B)$ with the Calkin algebra C. In this setting, suppose that we have two extensions, with associated Busby invariants τ_1, τ_2: $A \to C$. Then we can form the direct sum $\tau_1 \oplus \tau_2$: $A \to C \oplus C$. But since $H \oplus H \cong H$, we can embed $C \oplus C$ in C, so that $\tau_1 \oplus \tau_2$ becomes a map from A to C and is therefore the Busby invariant of an extension. In this way the set of extensions of A by K (or of K by A if you prefer: the terminology is not standardized) becomes a commutative semigroup.

In the 1970s Brown, Douglas, and Fillmore made an intensive study of this semigroup for the case $A = C(X)$, X a compact metric space. They showed that it is in fact a group Ext(X), with many pleasant properties such as homotopy invariance and Bott periodicity. One of the most important features of this BDF theory is its connection with K-theory. The topological K-theory for a manifold is a generalized cohomology theory. Topologists had sought for a long time to find a concrete realization for the associated K-homology, and it was known to Atiyah and others that this would involve the index of certain Fredholm operators. Brown, Douglas, and Fillmore succeeded in identifying $\mathrm{Ext}(X)$ with the K-homology group $K_1(X)$.

In seeking to extend this theory to more general classes of extensions, we look next at the case where the quotient algebra A is any C^*-algebra, the ideal still being K. The situation here is not so good, since in general $\mathrm{Ext}(A)$ is not a group (it can contain elements with no inverse). If A is separable and nuclear, however, then $\mathrm{Ext}(A)$ is a group. So let us press ahead and consider the set $\mathrm{Ext}(A, B)$ of extensions of A by B, for arbitrary C^*-algebras A and B. The first problem here is that we cannot even define a semigroup operation on $\mathrm{Ext}(A, B)$ unless B shares some of the "absorbent" properties of K which enabled us to handle direct sums of Busby invariants. This is not a serious difficulty. All that is needed is that B should be *stable*, that is, $B \cong B \otimes K$. If B is not stable then we can stabilize it by replacing it with $B \otimes K$, which is always stable. It then turns out that $\mathrm{Ext}(A, B \otimes K)$ is a group for any B, provided that A is separable and nuclear. If we assume in addition that B satisfies a mild separability condition then $\mathrm{Ext}(A, B \otimes K)$ is homotopy-invariant for both variables A, B and satisfies Bott periodicity.

We now find ourselves looking at bifunctors from pairs of C^*-algebras to abelian groups, and the scene is set for KK-theory. This was introduced by Kasparov, and gives a unified framework for all the material discussed above, as well as introducing some very powerful new machinery. What Kasparov did was to associate with each pair of C^*-algebras A, B an abelian group $KK(A, B)$ with many good properties (these usually require some kind of separability hypotheses, which we omit in what follows). For a start, $KK(\mathbf{C}, B) \cong K_0(B)$, so that KK-theory subsumes K-theory. The groups $KK(SA, B)$ and $KK(A, SB)$ are isomorphic, and are usually written as $KK^1(A, B)$. Then $KK^1(A, B)$ can be identified with the group of invertible elements of $\mathrm{Ext}(A, B \otimes K)$, and in particular is isomorphic to $\mathrm{Ext}(A, B)$ if A is nuclear and B is stable. The KK-groups are homotopy invariant and satisfy Bott periodicity in both variables. At a technical level, the usefulness of the KK-groups centers round the existence of the Kasparov product. This product is a map

$$KK(A, D) \times KK(D, B) \to KK(A, B)$$

which exists for any C^*-algebras A, D, B and which is associative. Both the construction of the product and the proof of associativity are technically difficult and can also seem conceptually obscure, especially in Kasparov's original formulation. There is an alternative formulation, due to Cuntz, in which elements of $KK(A, B)$ are viewed as generalized homomorphisms from A to B. The Kasparov product then appears as an analogue of the composition of homomorphisms. This approach can seem more accessible than Kasparov's; but it is less well adapted to some of the more important applications in analysis. In Kasparov's approach, $KK(A, B)$ is defined in terms of graded operators on a $\mathbf{Z}_2$-graded Hilbert B-module which commute modulo compact operators with an action of A on the module. (As in the case of extensions, there are several different possible choices for an equivalence relation on these objects, most of which eventually turn out to be the same.) This machinery can seem intimidating at first, but in fact such objects often arise in analysis. For example, certain types of maps between smooth manifolds give rise to a "Dirac operator", an elliptic pseudodifferential operator which fits very naturally into this framework.

To see how the Kasparov product can be a useful device, consider what happens when $A = \mathbf{C}$. Then a given element of the Kasparov group $KK(D, B)$ provides, via the Kasparov product, a homomorphism from $K_0(D)$ to $K_0(B)$. This provides a useful and natural way for constructing homomorphisms between K-groups. The Kasparov product also exhibits a duality between $K_0(B) = KK(\mathbf{C}, B)$ and $KK(B, \mathbf{C})$, thus extending the BDF theory in describing K-homology. Notice also that the Kasparov product makes $KK(A, A)$ into a ring. This ring structure is not very useful, however. Indeed, there is a Universal Coefficient Theorem which says among other things that, at least for a wide class of C^*-algebras, the ring $KK(A, A)$ can be reconstructed from the groups $K_0(A)$ and $K_1(A)$.

All the above material and much more is contained in the book under review. Blackadar's style is extremely concise. This enables him to pack an enormous amount of mathematics into just over 300 pages, but it means that many details are left for the reader to supply. This approach has its advantages. By exhibiting the bare bones of an argument, the author is able to clarify its essential structure in a way which a more detailed exposition might not achieve. It is important in such an approach that there should be no cheating: the details which are suppressed must be more or less routine, and the steps which are highlighted must be the key ones. Blackadar cannot be faulted on this test. For example, the proof of Bott periodicity occupies just two and a half pages, and within that space the essence of a long argument is conveyed with admirable clarity. As the book progresses, however, the tendency towards brevity intensifies. By Chapter 20 (*Equivariant KK-theory*) the author is forced to admit that "we must content ourselves with a survey".

Despite the terseness of the proofs, this is not a difficult book to read. The author takes great care to orient the reader and motivate the material at each stage, and there are very many illuminating comments, examples and exercises. A noteworthy feature is the careful presentation of apparently trivial examples where these provide useful insights. (See for example the discussion of $KK(\mathbf{C}, \mathbf{C})$ in Example 17.3.4.)

What is operator algebraic K-theory good for, and where does it go from here? Blackadar answers the first of these questions in a final chapter (*Survey of applications to geometry and topology*) which discusses a variety of existing and potential uses for K-theory and KK-theory. These include index theorems, the Novikov Conjecture on higher signatures, Rosenberg's work on manifolds with positive scalar curvature and the Baum-Connes Conjecture. As for the second question, suppose that we ask that the commutators which occur in Kasparov's definition for the elements of $KK(A, \mathbf{C})$ should not just be compact but should lie in one of the Schatten p-classes. One can then exploit the fact that certain operators are of trace class ("one" here means not just anyone but, to be precise, Connes) and one can show that the trace behaves like a noncommutative version of the Chern character. This opens up a whole new subject of "noncommutative differential geometry". Furthermore, the algebraic formalism of the behavior of the trace leads one to the theory of cyclic cohomology. "But that is the subject for another book [**Cn 3**]", as Blackadar says at the end of his final chapter. (If you can't guess what the "[**Cn 3**]" refers to then you will have to look it up in Blackadar's bibliography.)

Final verdict: this is an excellent book, combining formidable scholarship, impeccable accuracy, and lucid if succinct exposition. It sets a very high standard for Springer's commendable new series of MSRI Publications.

CHRISTOPHER LANCE

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Applications of Lie groups to differential equations, by Peter J. Olver. Graduate Texts in Mathematics, Volume 107, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1986, xxvi + 497 pp., $54.00. ISBN 0-387-96250-6

A standard introductory textbook on ordinary or partial differential equations presents the student with a maze of seemingly unrelated techniques to construct solutions. Usually these unmotivated and boring techniques constitute the total experience with differential equations for an undergraduate. Faced with a given differential equation which is not a textbook model, one is hopelessly lost without "hints"!

In the latter part of the 19th century Sophus Lie introduced the notion of continuous groups, now known as Lie groups, in order to unify and extend these bewildering special methods, especially for ordinary differential equations. Lie was inspired by lectures of Sylow given at Christiania, present-day Oslo, on Galois theory and Abel's related works. [In 1881 Sylow and Lie collaborated in editing the complete works of Abel.] He aimed to use symmetry to connect the various solution methods for ordinary differential equations in the spirit of the classification theory of Galois and Abel for polynomial equations. Lie showed that the order of an ordinary differential equation can be reduced by one if it is invariant under a one-parameter Lie group of point transformations. His procedures were both constructive and aesthetic.

For ordinary differential equations Lie's work systematically and comprehensibly related a miscellany of topics including: integrating factors, separable equations, homogeneous equations, reduction of order, the method of undetermined coefficients, the method of variation of parameters, Euler equations, and homogeneous equations with constant coefficients. Lie also indicated that