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AN EXTENSION OF THE KAHANE-KHINCHINE INEQUALITY

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Let $\omega_1, \ldots, \omega_2, \ldots,$ denote the *Steinhaus variables*: independent identically distributed random variables, uniformly distributed on [0,1].

THEOREM. *There exists $c > 0$ such that if $x_1, \ldots, x_N$ are elements of any (complex) Banach space B then*

$$\exp \mathsf{E} \log \left\| \sum_{j=1}^{N} e^{2\pi i \omega_j} x_j \right\| \geq c \left\{ \mathsf{E} \left\| \sum_{j=1}^{N} e^{2\pi i \omega_j} x_j \right\|^2 \right\}^{1/2}. \tag{1}$$

(Here "exp" is the exponential, and "E" denotes "expected value".)

The Kahane-Khinchine inequality (see [**KH**, Chapter 2, Theorem 4, or **AG**, p. 176] for the original proof; another argument due to C. Borell may be found in [**BK** or **LT**, Theorem 1.e.13]) states that

$$\left\{ \mathsf{E} \left\| \sum_{j=1}^{N} e^{2\pi i \omega_j} x_j \right\|^p \right\}^{1/p} \geq c_p \left\{ \mathsf{E} \left\| \sum_{j=1}^{N} e^{2\pi i \omega_j} x_j \right\|^2 \right\}^{1/2} \qquad (p > 0). \tag{2}$$

Recalling that in general $\{\mathsf{E}|f|^p\}^{1/p}$ decreases to $\exp \mathsf{E} \log |f|$ as p decreases to zero, one sees that (1) is a strictly stronger statement than (2); in fact (1) says simply that c_p may be taken bounded away from zero in (2). Note that the inequality obtained from (1) by replacing $e^{2\pi i \omega_j}$ with the jth Rademacher function r_j is false, even in the case $B = \mathbf{C}$: If $s_n = n^{-1/2}(r_1 + \cdots + r_n)$ then $\exp \mathsf{E} \log |s_n| = 0$ for even values of n, although s_n is asymptotically normal. In other words: Suppose that X is a random variable; suppose even $|X| \leq 1$ a.s. Then to say $\exp \mathsf{E} \log |X| \geq c$ implies that the set where X is small must be small, while to say $\{\mathsf{E}|X|^p\}^{1/p} \geq c$ does not even preclude the possibility that X vanish on a set of positive measure!

In the case $B = \mathbf{C}$ inequality (1) is proved in [**UK**], and various applications are given. In particular one may use (1) to show that the zero set of a Bloch function may be strictly larger than is possible for a function in the "little-oh" Bloch space, answering a question of Ahern and Rudin [**AR**]; this fact then gives a result analogous to Theorem 6.1 of [**AR**], with VMOA and H^∞ replaced by BMOA and VMOA, respectively. Inequality (1) also allows one to construct new and improved Ryll-Wojtaszczyk polynomials [**RW**]: There exists a sequence $P_1, P_2, \ldots,$ of polynomials in $\mathbf{C}^n$ such that P_j is homogeneous of degree j and satisfies $|P_j(z)| \leq 1$ $(z \in \mathbf{C}^n, |z| \leq 1)$ while

$$\exp \int_S \log |P_j| \, d\sigma \geq c > 0. \tag{3}$$

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Here σ is normalized Lebesgue measure on S, the unit sphere in $\mathbf{C}^n$. (Ryll and Wojtaszczyk give

$$\int_S |P_j|^2\, d\sigma \geq c \tag{4}$$

in place of (3).)

Now let $\tilde{\omega}_1, \tilde{\omega}_2, \ldots,$ be a second sequence of Steinhaus variables, independent of the $\omega_1, \omega_2, \ldots$. It is not too difficult to see that our theorem implies

$$\exp \mathsf{E} \log \left\| \sum_{j,k=1}^{N} e^{2\pi i \omega_j} e^{2\pi i \tilde{\omega}_k} x_{j,k} \right\| \geq c \left\{ \mathsf{E} \left\| \sum_{j,k=1}^{N} e^{2\pi i \omega_j} e^{2\pi i \tilde{\omega}_k} x_{j,k} \right\|^2 \right\}^{1/2} \tag{5}$$

for $x_{j,k} \in B$ $(1 \leq j, k \leq N)$. (And similarly for n mutually independent sequences of Steinhaus variables, by induction.) The proof involves applying (1) in a certain space of square-integrable B-valued random variables; thus it would appear that even the special case of (5) corresponding to $B = \mathbf{C}$ does not follow directly from results in [**UK**], but rather constitutes an application of the present "vector-valued" inequality to the scalar-valued case.

We would like to give an idea of the proof of (1):

Suppose that $\mathsf{E}\| \sum_{j=1}^{N} e^{2\pi i \omega_j} x_j \|^2 = 1$, and define

$$\Psi(\lambda) = P\left(\left\| \sum_{j=1}^{N} e^{2\pi i \omega_j} x_j \right\| < \lambda \right) \tag{6}$$

for $\lambda > 0$. We need only show that

$$\int_0^1 \Psi(\lambda) \frac{d\lambda}{\lambda} \leq c. \tag{7}$$

Take $\|x_1\| \geq \|x_j\|$ for all j. For $0 < \lambda < \|x_1\|/2$ the triangle inequality shows that

$$P(\|e^{2\pi i \omega_1} x_1 + y\| < \lambda) \leq c\lambda / \|x_1\|$$

for any $y \in B$; this gives

$$\int_0^{\|x_1\|/2} \Psi(\lambda) \frac{d\lambda}{\lambda} \leq c, \tag{8}$$

by independence. Since $\Psi(\lambda) \leq 1$, (8) leads to

$$\int_0^{K\|x_1\|} \Psi(\lambda) \frac{d\lambda}{\lambda} \leq c \tag{9}$$

for any fixed K. Inspired by Theorem 3 in Chapter 2 of [**KH**] (or see inequality 2.5 on p. 106 of [**AG**]) we were able to prove a sort of "concentration inequality":

LEMMA. *If K is large enough then there exists $\gamma \in (0,1)$ such that if $K\|x_1\| \leq \lambda \leq 1$ then $\Psi(\gamma\lambda) \leq \frac{1}{2}\Psi(\lambda)$.*

It is easy to see that the lemma implies

$$\int_{K\|x_1\|}^1 \Psi(\lambda) \frac{d\lambda}{\lambda} \leq c; \tag{10}$$

certainly (9) and (10) give (7). □

Note that in the case $B = \mathbf{C}$ one may use the Fourier transform (the "method of characteristic functions") to establish (10) more simply; see [**UK**].

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