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SINGULAR LOCI OF SCHUBERT VARIETIES FOR CLASSICAL GROUPS

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In this note, we give an explicit description of the singular locus of a Schubert variety in the flag variety G/B, where G is a classical group, and B a Borel subgroup of G.

Let G be a classical group, and T a maximal torus in G. Let W be the Weyl group, and R the system of roots, of G relative to T. Let B be a Borel subgroup of G, where $B \supset T$. Let S (resp. R^+) be the set of simple (resp. positive) roots of R relative to B. For $\alpha \in R$, let s_α be the reflection with respect to α, and X_α the element in the Chevalley basis for the Lie algebra of G, associated to α. For $w \in W$, let $e(w)$ denote the point in G/B corresponding to w. The Schubert variety $X(w)$, where $w \in W$, is by definition the Zariski closure of $B\ e(w)$ in G/B. ($X(w)$ is understood to be endowed with the canonical reduced structure.) Let $\succeq$ denote the Bruhat order in W. It is well known that for $w_1, w_2 \in W$,

$$w_1 \succeq w_2 \quad \text{if and only if} \quad X(w_1) \supseteq X(w_2).$$

(For generalities on algebraic groups, one may refer to [**1**].)

The results on the singular locus of a Schubert variety are obtained as consequences of "standard monomial theory" as developed in *Geometry of* G/P. I–V (cf. [**11, 7, 4, 5, 8**]). One of the consequences of standard monomial theory is the First Basis Theorem (cf. [**5, 8, 6**]) which gives a $\mathbf{Z}$ basis

$$\{P(\lambda,\mu),\ (\lambda,\mu) \text{ an admissible pair}\}$$

for $H^0(G_{\mathbf{Z}}/P_{\mathbf{Z}}, L_{\mathbf{Z}})$, where $P_{\mathbf{Z}}$ is a maximal parabolic subgroup scheme of $G_{\mathbf{Z}}$ and $L_{\mathbf{Z}}$ is the ample generator of $\operatorname{Pic}(G_{\mathbf{Z}}/P_{\mathbf{Z}})$. For any field k, let us denote the canonical image of $P(\lambda,\mu)$ in $H^0(G_{\mathbf{Z}}\otimes k/P_{\mathbf{Z}}\otimes k, L_{\mathbf{Z}}\otimes k)$ by $p(\lambda,\mu)$. In [**9**], it is shown that over any field k, for $w,\tau \in W$, with $w \succeq \tau$, the Zariski tangent space $T(w,\tau)$, to $X(w)$ at $e(\tau)$ is spanned by

$$\left\{ X_{-\beta}, \beta \in \tau(R^+) \;\middle|\; \begin{array}{l} \text{for all } (\lambda,\mu) \text{ such that } X_{-\beta}p(\lambda,\mu) = cp(\tau,\tau), c \in k^*, \\ p(\lambda,\mu)|_{X(w)} \not\equiv 0 \end{array} \right\}.$$

Denoting by $\{Q(\lambda,\mu)\}$ the basis for the $\mathbf{Z}$-dual of $H^0(G_{\mathbf{Z}}/P_{\mathbf{Z}}, L_{\mathbf{Z}})$, dual to the basis $\{P(\lambda,\mu)\}$, it can be seen easily that $X_{-\beta}p(\lambda,\mu) = cp(\tau,\tau)$, $c \in k^*$, if and only if $X_{-\beta}Q(\tau,\tau)$, when written as a $\mathbf{Z}$-linear combination of the elements

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$Q(\theta,\delta)$, involves $Q(\lambda,\mu)$ with a coefficient that is nonzero in k. From this we obtain that $T(w,\tau)$ is spanned by

$$\left\{ X_{-\beta}, \beta\in\tau(R^+) \;\middle|\; \begin{array}{l}\text{for all } (\lambda,\mu) \text{ such that } Q(\lambda,\mu) \text{ occurs in } X_{-\beta}Q(\tau,\tau)\\ \text{with a coefficient that is nonzero in } k, w\succeq\lambda\end{array} \right\}.$$

In [3], we have given an explicit description of $Q(\lambda,\mu)$ for the case of a classical group. Using this description, we express $X_{-\beta}Q(\tau,\tau)$ as a linear combination with integer coefficients of the $Q(\theta,\delta)$'s. This enables us to obtain an explicit description of the singular locus of $X(w)$.

Let G be classical of rank n. Let $S=\{\alpha_1,\dots,\alpha_n\}$, the order being as in [2]. Further, we follow the notation in [2] to denote the elements of R. For $1\le d\le n$, we fix the following:

$$\begin{aligned} P_d &= \begin{cases}\text{the maximal parabolic subgroup of } G\\ \text{obtained by ``omitting the simple root } \alpha_d\text{''},\end{cases}\\ W_{P_d} &= \text{Weyl group of } P_d,\\ W^{P_d} &= \text{the set of ``minimal representatives'' of } W/W_{P_d}. \end{aligned}$$

Recall (cf. [2, 4]) that

$$W^{P_d}=\{w\in W\mid l(ws_{\alpha_i})=l(w)+1,\ 1\le i\le n,\ i\ne d\}.$$

It is known (cf. [2]) that

$$W=W_{P_d}\cdot W^{P_d}. \tag{1}$$

For $w\in W$, let $w^{(d)}$ be the element in W^{P_d} corresponding to the coset wW_{P_d}. We have

$$w^{(d)}=wW_{P_d}\cap W^{P_d}. \tag{2}$$

Let

$$A=\{(a_1,\dots,a_d)\mid a_1<a_2<\cdots<a_d,\ a_i\in\mathbf{Z}\}.$$

We have a natural partial order $\ge$ in A, namely,

$$(a_1,\dots,a_d)\ge(b_1,\dots,b_d),\quad \text{if } a_i\ge b_i,\ 1\le i\le d. \tag{3}$$

This partial order among d-tuples will be used in the sequel in describing the Bruhat order in W^{P_d}. Further, for any d-tuple $(z_1,\dots,z_d)$ of integers, we let

$$(z_1,\dots,z_d)\uparrow=(z_{i_1},z_{i_2},\dots,z_{i_d}) \tag{4}$$

where $j\to i_j$ is a permutation and $z_{i_j}\le z_{i_{j+1}}$. Thus, $(z_1,\dots,z_d)\uparrow$ is the d-tuple whose entries are obtained by arranging the entries $(z_1,\dots,z_d)$ in increasing order. We shall denote the elements of the symmetric group S_m, where $m\in\mathbf{N}$, in the following way. Let $\sigma\in S_m$ be such that

$$\sigma(i)=c_i,\qquad 1\le i\le m. \tag{5}$$

We shall denote σ by $(c_1\cdots c_m)$. Let k be the base field. For any positive integer m, let $\{e_1,\dots,e_m\}$ denote the standard basis of k^m.

I. The symplectic group $\mathrm{Sp}(2n)$. Let $E = \left(\begin{smallmatrix} 0 & J \\ -J & 0 \end{smallmatrix}\right)$, where

$$J = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}_{n\times n}.$$

Let (,) be the skew symmetric bilinear form on k^{2n}, represented by E, with respect to $\{e_1, \dots, e_{2n}\}$. Let

$$G = \mathrm{Sp}(2n) = \{A \in \mathrm{SL}(2n) \mid {}^tAEA = E\}. \tag{6}$$

Let σ be the involution on $\mathrm{SL}(2n)$ defined by

$$\sigma(A) = E({}^tA)^{-1}E^{-1}, \qquad A \in \mathrm{SL}(2n). \tag{7}$$

We see that

$$\mathrm{Sp}(2n) = \mathrm{SL}(2n)^{\sigma}. \tag{8}$$

In view of (8), we obtain an identification of W, the Weyl group of G, with a subgroup of S_{2n} (= the Weyl group of $\mathrm{SL}(2n)$), namely

$$W = \{(a_1 \cdots a_{2n}) \mid a_i = 2n + 1 - a_{2n+1-i},\ 1 \le i \le 2n\}. \tag{9}$$

See [7] for details.

The above identification (cf. (9)) of W, and straightforward calculations using the definitions of [2] allow us to identify W^{P_d} as

$$W^{P_d} = \left\{ (a_1, \dots, a_d) \;\middle|\; \begin{array}{l} (1)\ 1 \le a_1 < a_2 < \cdots < a_d \le 2n, \\ (2) \text{ for } 1 \le i \le 2n, \text{ if } i \in \{a_1, \dots, a_d\}, \\ \text{then } 2n + 1 - i \notin \{a_1, \dots, a_d\} \end{array} \right\}. \tag{10}$$

For $w \in W$, say $w = (c_1 \cdots c_{2n})$, we see easily that

$$w^{(d)} = (c_1, \dots, c_d) \uparrow. \tag{11}$$

Under the above identification of W^{P_d}, we have (cf. [10]), given two elements $(a_1, \dots, a_d)$, $(b_1, \dots, b_d)$ in W^{P_d},

$$(a_1, \dots, a_d) \succeq (b_1, \dots, b_d) \quad \text{if and only if} \quad (a_1, \dots, a_d) \ge (b_1, \dots, b_d). \tag{12}$$

Thus, the Bruhat order in W^{P_d} coincides with the natural order (cf. equation (3)) on d-tuples.

PROPOSITION C.1. *Let $G = \mathrm{Sp}(2n)$. For $1 \le i \le 2n$, let $i' = 2n + 1 - i$ and $|i| = \min\{i, i'\}$. Let $w, \tau \in W$, with $w \succeq \tau$. Let $\tau = (a_1 \cdots a_{2n})$. Then the tangent space $T(w, \tau)$ to $X(w)$ at $e(\tau)$ is spanned by the set of root vectors $\{X_{-\beta}, \beta \in N(w, \tau)\}$, where $N(w, \tau)$ is the subset of $\tau(R^+)$ consisting of roots β which satisfy criteria* (a) *and* (b) *below. Let $\beta = \tau(\ a)$, $\alpha \in R^+$. We follow the notation of* [2] *for elements of R^+.*

(a) *Let $\alpha = \varepsilon_j - \varepsilon_k$, $1 \le j < k \le n$ or $2\varepsilon_j$, $1 \le j \le n$. Then*

$$w \succeq s_\beta \tau.$$

(b) *Let* $\alpha = \varepsilon_j + \varepsilon_k$, $1 \le j < k \le n$. *Let* s (*resp.* r) *be the* $\min\{|a_j|, |a_k|\}$ (*resp.* $\max\{|a_j|, |a_k|\}$). *Then*

$$w^{(j)} \succeq (a_1, \ldots, a_{j-1}, a'_k) \uparrow$$

and

$$w^{(k)} \succeq (a_1, \ldots, \hat{a}_j, \ldots, a_{k-1}, r, s') \uparrow .$$

II. The special orthogonal group $\mathrm{So}(2n+1)$. Let

$$E = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}_{2n+1 \times 2n+1},$$

and let (,) be the symmetric bilinear form on k^{2n+1}, respresented by E, with respect to $\{e_1, \ldots, e_{2n+1}\}$. Let

(13) $$G = \mathrm{So}(2n+1) = \{A \in \mathrm{SL}(2n+1) \mid {}^t AEA = E\}.$$

Let σ be the involution on $\mathrm{SL}(2n+1)$ defined by

(14) $$\sigma(A) = E({}^t A)^{-1} E, \qquad A \in \mathrm{SL}(2n+1).$$

As in §I, we have

(15) $$\mathrm{So}(2n+1) = \mathrm{SL}(2n+1)^{\sigma}.$$

In view of (15), we obtain identifications for the Weyl group W, and also for W^{P_d} similar to (9) and (10), namely

(16) $$W = \{(a_1 \cdots a_{2n+1}) \in S_{2n+1} \mid a_i = 2n+2 - a_{2n+2-i},\ 1 \le i \le 2n+1\}$$

and

(17) $$W^{P_d} = \left\{ (a_1, \ldots, a_d) \;\middle|\; \begin{array}{l} (1)\ 1 \le a_1 < a_2 < \cdots < a_d \le 2n+1, \\ (2)\ a_i \ne n+1, 1 \le i \le d, \\ (3)\ \text{For } 1 \le i \le 2n+1, \text{ if } i \in \{a_1, \ldots, a_d\} \\ \textit{then } 2n+2-i \notin \{a_1, \ldots, a_d\} \end{array} \right\}.$$

For $w \in W$, say $w = (c_1 \cdots c_{2n+1})$, we have

(18) $$w^{(d)} = (c_1, \ldots, c_d) \uparrow .$$

As in §I, we have (cf. [**10**]) that the Bruhat order in W^{P_d} coincides with the natural order (cf. equation (3)) on d-tuples.

PROPOSITION B.1. (*Assume* $\mathrm{char}\, k \ne 2$.) *Let* $G = \mathrm{So}(2n+1)$. *For* $1 \le i \le 2n+1$, *let* $i' = 2n+2-i$ *and* $|i| = \min\{i, i'\}$. *Let* $w, \tau \in W$, *with* $w \succeq \tau$, *and let* $\tau = (a_1 \cdots a_{2n+1})$. *Then the tangent space* $T(w,\tau)$ *to* $X(w)$ *at* $e(\tau)$ *is spanned by the set of root vectors* $\{X_{-\beta},\ \beta \in N(w,\tau)\}$, *where* $N(w,\tau)$ *is the subset of* $\tau(R^+)$ *consisting of roots* β *which satisfy criteria* (a), (b), *and* (c) *below. Let* $\beta = \tau(\alpha)$, $\alpha \in R^+$.

(a) *Let* $\alpha = \varepsilon_j - \varepsilon_k$, $1 \le j < k \le n$. *Then*

$$w \succeq s_\beta \tau.$$

(b) *Let* $\alpha = \varepsilon_j + \varepsilon_k$, $1 \le j < k \le n$. *Let* s (*resp.* r) *be the minimum* (*resp. maximum*) *of* $\{|a_j|, |a_k|\}$.

(i) *Suppose precisely one of* $\{a_j, a_k\}$ *does not exceed* n. *Then*

$$w^{(j)} \succeq (a_1, \ldots, a_{j-1}, a'_k) \uparrow,$$
$$w^{(k)} \succeq (a_1, \ldots, \hat{a}_j, \ldots, a_{k-1}, r, s') \uparrow,$$

and

$$w^{(n)} \succeq (s_\beta \tau)^{(n)}.$$

(ii) *Suppose* a_j, a_k *either both exceed* n *or both do not exceed* n. *For* $k \le d \le n-1$, *let* $s_{c(d)}$ *be the largest integer*, $r < s_{c(d)} \le n$, *such that* $s_{c(d)} \notin \{|a_1|, \ldots, |a_d|\}$ (*if no such integer exists, we let* $s_{c(d)} = r$). *Then*

$$w^{(j)} \succeq (a_1, \ldots, a_{j-1}, a'_k) \uparrow,$$
$$w^{(d)} \succeq (a_1, \ldots, \hat{a}_j, \ldots, \hat{a}_k, \ldots, a_d, s'_{c(d)}, s') \uparrow, \qquad k \le d \le n-1,$$

and

$$w^{(n)} \succeq (s_\beta \tau)^{(n)}.$$

(c) *Let* $\alpha = \varepsilon_j$, $1 \le j \le n$. *For* $j \le d \le n-1$, *let* $s_{m(d)}$ *be the largest integer*, $|a_j| < s_{m(d)} \le n$, *such that* $s_{m(d)} \notin \{|a_1|, \ldots, |a_d|\}$ (*if no such* $s_{m(d)}$ *exists, we let* $s_{m(d)} = |a_j|$). *Then*

$$w^{(d)} \succeq (a_1, \ldots, \hat{a}_j, \ldots, a_d, s'_{m(d)}) \uparrow, \qquad j \le d \le n-1,$$

and

$$w^{(n)} \succeq (s_\beta \tau)^{(n)}.$$

III. The special orthogonal group So$(2n)$. Let

$$E = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}_{2n \times 2n},$$

and let (,) be the symmetric bilinear form on k^{2n}, represented by E, with respect to $\{e_1, \ldots, e_{2n}\}$. Let

$$G = \mathrm{So}(2n) = \{A \in \mathrm{SL}(2n) \mid {}^t A E A = E\}. \tag{19}$$

Let σ be the involution on SL$(2n)$ defined by

$$\sigma(A) = E({}^t A)^{-1} E, \qquad A \in \mathrm{SL}(2n). \tag{20}$$

We have

$$\mathrm{So}(2n) = \mathrm{SL}(2n)^\sigma. \tag{21}$$

As in §§I and II, we obtain, in view of (21), identifications (described below) for W and W^{P_d}. We have

$$W = \left\{ (a_1 \cdots a_{2n}) \in S_{2n} \;\middle|\; \begin{array}{l} (1)\ a_i = 2n+1-a_{2n+1-i},\ 1 \le i \le 2n, \\ (2)\ \#\{i,\ 1 \le i \le n \mid a_i > n\} \text{ is even} \end{array} \right\}. \tag{22}$$

For $1 \leq d \leq n$, let

$$(23)\quad Z_d = \left\{ (a_1, \dots, a_d) \;\middle|\; \begin{array}{ll} (1) & 1 \leq a_1 < a_2 < \cdots < a_d \leq 2n, \\ (2) & \text{for } 1 \leq i \leq 2n, \text{ if } i \in \{a_1, \dots, a_d\}, \text{ then} \\ & 2n+1-i \notin \{a_1, \dots, a_d\} \end{array} \right\}.$$

We have for $d \neq n-1$

$$(24)\qquad W^{P_d} = Z_d.$$

For $d = n-1$, if $w \in W^{P_d}$, then

$$(25)\qquad w \equiv wu_i \ (\operatorname{mod} W_{P_{n-1}}), \qquad 0 \leq i \leq n,\ i \neq n-1,$$

where

$$(26)\qquad u_i = \begin{cases} s_{\alpha_n} & \text{if } i = n, \\ \mathrm{Id} & \text{if } i = 0, \\ s_{\alpha_i} s_{\alpha_{i+1}} \cdots s_{\alpha_{n-2}} s_{\alpha_n} & \text{if } 1 \leq i \leq n-2. \end{cases}$$

(Here Id denotes the identity element in W.) In particular, for $w_1, w_2 \in W$, say $w_1 = (a_1 \cdots a_{2n})$, $w_2 = (b_1 \cdots b_{2n})$, we can have $w_1^{(n-1)} = w_2^{(n-1)}$ without $(a_1, \dots, a_{n-1}) \uparrow$ and $(b_1, \dots, b_{n-1}) \uparrow$ being the same. Thus $W^{P_{n-1}}$ gets identified with a *proper* subset of Z_{n-1} (cf. Definition (23)). For $w \in W$, say $w = (c_1 \cdots c_{2n})$, we have

$$(27)\qquad w^{(d)} = (c_1, \dots, c_d) \uparrow, \qquad d \neq n-1.$$

To describe $w^{(n-1)}$, we let, for $1 \leq i \leq n$, $i \neq n-1$,

$$(28)\qquad (y_1^{(i)}, \dots, y_{n-1}^{(i)}) = \begin{cases} \text{the } (n-1)\text{-tuple given by the first } (n-1) \\ \text{entries in } wu_i \end{cases}$$

and

$$(29)\qquad Y = \{(y_1^{(i)}, \dots, y_{n-1}^{(i)}) \uparrow,\ 0 \leq i \leq n,\ i \neq n-1\}.$$

We observe that Y is totally ordered under $\geq$ (cf. (3)). We have

$$(30)\qquad w^{(n-1)} = \text{the smallest (under } \geq \text{) element in } Y.$$

Unlike the cases of $\mathrm{Sp}(2n)$ (resp. $\mathrm{So}(2n+1)$), the Bruhat order in W, the Weyl group of $\mathrm{So}(2n)$, is not induced from the Bruhat order in S_{2n}. Hence the Bruhat order in W^{P_d} does not coincide with the natural order on d-tuples (cf. (3)). We now describe the Bruhat order in W^{P_d}.

For $1 \leq i \leq 2n$, let

$$i' = 2n+1-i \quad \text{and} \quad |i| = \min\{i, i'\}.$$

Under the above identification, given two elements $(a_1, \dots, a_d)$, $(b_1, \dots, b_d)$ in W^{P_d}, $1 \leq d \leq n$, we have (cf. [**10**])

$$(a_1, \dots, a_d) \succeq (b_1, \dots, b_d)$$

if and only if the following two conditions hold:

(A) $(a_1, \dots, a_d) \geq (b_1, \dots, b_d)$.

(B) Suppose for some r, $1 \le r \le d$, and some i, $0 \le i \le d-r$,

$$(|a_{i+1}|, \dots, |a_{i+r}|) \uparrow = (|b_{i+1}|, \dots, |b_{i+r}|) \uparrow = \{n+1-r, \dots, n\}.$$

Then

$$\#\{j,\ i+1 \le j \le i+r \mid a_j > n\}$$

and

$$\#\{j,\ i+1 \le j \le i+r \mid b_j > n\}$$

should both be odd or both even.

PROPOSITION D.1. (*Assume* char $k \neq 2, 3$.) *Let* $G = \mathrm{So}(2n)$. *Let* $w, \tau \in W$, *with* $w \succeq \tau$, *and let* $\tau = (a_1 \cdots a_{2n})$. *Then the tangent space* $T(w,\tau)$ *to* $X(w)$ *at* $e(\tau)$ *is spanned by the set of root vectors* $\{X_{-\beta}, \beta \in N(w,\tau)\}$, *where* $N(w,\tau)$ *is the subset of* $\tau(R^+)$ *consisting of roots* β *which satisfy criteria* (a) *and* (b) *below. Let* $\beta = \tau(\alpha)$, $\alpha \in R^+$.

(a) *Let* $\alpha = \varepsilon_j - \varepsilon_k$, $1 \le j < k \le n$. *Then*

$$w \succeq s_\beta \tau.$$

(b) *Let* $\alpha = \varepsilon_j + \varepsilon_k$, $1 \le j < k \le n$. *Let* s *(resp.* r*) be the minimum (resp. maximum) of* $\{|a_j|, |a_k|\}$.

(i) *Suppose precisely one of* $\{a_j, a_k\}$ *does not exceed* n. *Then*

$$\begin{aligned} w^{(j)} &\succeq (a_1, \dots, a_{j-1}, a_k') \uparrow, \\ w^{(k)} &\succeq (a_1, \dots, \hat{a}_j, \dots, a_{k-1}, r, s') \uparrow, \\ w^{(n-1)} &\succeq (s_\beta \tau)^{(n-1)}, \end{aligned}$$

and

$$w^{(n)} \succeq (s_\beta \tau)^{(n)}.$$

(ii) *Suppose* a_j, a_k *either both exceed* n *or both do not exceed* n. *For* $k \le d \le n-2$, *let* $s_{-l(d)}, \dots, s_{-1}, s_0, s_1, \dots, s_{c(d)}$ *be the integers*

$$s < s_{-l(d)} < s_{-l(d)+1} < \cdots < s_{-1} < s_0 = r < s_1 < \cdots < s_{c(d)} \le n$$

such that

$$s_i \notin \{|a_1|, \dots, |a_d|\}, \qquad -l(d) \le i \le c(d), \quad i \neq 0.$$

Then

$$\begin{aligned} w^{(j)} &\succeq (a_1, \dots, a_{j-1}, a_k') \uparrow, \\ w^{(d)} &\succeq \begin{cases} (a_1, \dots, \hat{a}_j, \dots, \hat{a}_k, \dots, a_d, s'_{c(d)-1}, s') \uparrow & \text{if } (l(d), c(d)) \neq (0,0), \\ (a_1, \dots, \hat{a}_j, \dots, \hat{a}_k, \dots, a_d, r', s') \uparrow & \text{if } (l(d), c(d)) = (0,0), \end{cases} \end{aligned}$$

and for $d = n-1$ *or* n,

$$w^{(d)} \succeq (s_\beta \tau)^{(d)}.$$

IV. Concluding remarks.

COROLLARY. *Let G be of type $\mathbf{B}_n$, $\mathbf{C}_n$, or $\mathbf{D}_n$ and let $w \in W$. Then $X(w)$ is smooth if and only if $\#N(w, \mathrm{Id}) = l(w)$, where $N(w, \mathrm{Id})$ is given by Proposition* C.1, B.1, *or* D.1 *according as G is of type C_n, B_n, or D_n, with $\tau = \mathrm{Id}$, the identity element of W.*

REMARK 1. For G of type A_n, similar results as above are described in [9].

REMARK 2. Even if char $k = 2$ or 3 (in the case of special orthogonal groups), using the explicit computations of $X_{-\beta}Q(\tau, \tau)$, one can still describe $T(w, \tau)$ in a way similar to Propositions B.1 and D.1.

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