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Amarts and set function processes, by Allen Gut and Klaus D. Schmidt, Lecture Notes in Mathematics, vol. 1042, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983, 258 pp., $12.50. ISBN 3-540-12867-0

What is an amart? Let (X_n) be a sequence of random variables adapted to increasing sigma algebras $\mathscr{L}_n$. A stopping time is a random variable T taking positive integer values and the value $+\infty$ such that if $n < \infty$, then the event $\{T = n\}$ (equivalently, $\{T \leqslant n\}$) is in $\mathscr{L}_n$. Intuitively, this means that the event T equals n is determined by the outcome of the trials up to the time n. If X_n is the fortune of a gambler at time n and the casino gives no credit, then the time when a ruined gambler must stop is also a stopping time in the mathematical sense. Stopping times, as propounded by J. L. Doob and, later, by the Strasbourg school led by P. A. Meyer, are among the most important features of modern probability. For convergence problems, of special importance are *simple* stopping times: those taking finitely many finite values. (Simple stopping times are also the only ones of practical importance: They do not require an infinite amount of time or an infinite fortune.) Let Σ be the collection of simple stopping times. A martingale can be defined by the property: the net $(EX_T\colon T \text{ in } \Sigma)$ is constant. If the same net *converges*, the process (X_n) is called an *amart* (originally an acronym for asymptotic martingale). Since amarts are to include martingales, it is crucial in this definition to allow only *simple* stopping times: Otherwise a martingale need not be an amart, as seen by the example of the famous original gambling martingale in which the player doubles his stake each time he loses. After the first article by John Baxter [5], the amart convergence theorem—asserting almost sure convergence of L_1-bounded amarts—was proved by D. G. Austin, G. A. Edgar, and A. Ionescu Tulcea (= A. Bellow) [3]. The same result was obtained in a less explicit but stronger form by R. V. Chacon [12]. Chacon's "Fatou's inequalities" were anticipated by W. D. Sudderth [40], who was influenced by L. Dubins, but Sudderth considered nonsimple stopping times. The amart convergence theorem, and more, was proved earlier in a measure-theoretical form involving *no* stopping times by C. Lamb [29]. We submit, however, that the amart theory started with simple stopping times, and that, in fact, the notion of simple stopping time is more basic here than the exact definition of the amart, since the deeper convergence theorems on directed sets have various forms, all of which share the use of simple stopping times. More about this later. The article of Chacon and the reviewer [13] initiates vector-valued amarts. These authors are the first to believe that the notion merits a name: *asymptotic martingale*. A systematic presentation of the amart theory paralleling the martingale theory, including for the first time the optional sampling theorem, the Riesz decomposition, the descending and the continuous parameter cases, was given by Edgar and the reviewer [21]. This article calls a spade a spade, introducing the term "amart". It was already obvious at that time that there would be many varieties of amarts, so that a short term was needed to leave room for adjectives.

What is interesting about amarts? First, a martingale is not an asymptotic notion, by which I mean that if the first few terms are changed, the martingale property is destroyed. Thus, clearly, the martingale property cannot be necessary and sufficient for convergence, but the amart property is, in the class of sequences with integrable supremum. Second, the class of L_1-bounded amarts is closed under lattice operations (equivalently, truncation); martingales, of course, are not. To be sure, there is an important asymptotic notion that is closed under lattice operations and generalizes the martingale: quasimartingale. Quasimartingales are essential in stochastic integration. Amarts include quasimartingales; to understand the difference, note that in the deterministic case (one-point space), amarts are exactly the sequences that converge; quasimartingales are exactly the sequences of bounded variation. If one is only interested in convergence, then the assumption of bounded variation is an overkill. The difference proved important in the theory of martingales indexed by the plane: Amart methods could solve open martingle problems for which the theory of the stochastic integral was not suitable (see A. Millet and the reviewer [37]).

Next is the matter of simplicity of proofs. A stopping time is a more elementary notion than conditional expectation in that its definition does not require the Radon-Nikodým theorem, and proofs involving stopping times are very intuitive. To make the point I will sketch the proof of a.s. convergence of amarts with the integrable supremum. The basic observation is that there is an increasing sequence of simple stopping times $T(n)$ such that $X_{T(n)}$ converges in probability to $X_* = \limsup X_n$. The reason for this is that lim sup—or any other accumulation point—manifests itself infinitely often on the way to infinity; it is like a light shining on the horizon. (This, of course, is not true for the supremum, which can all too easily be missed.) Thus, after we obtain $X_{T(1)}$ close in probability to X^*, we can find $T(2) > T(1)$ such that $X_{T(2)}$ is even closer to X^*, etc. Similarly, we obtain an increasing sequence of simple stopping times S_n such that $X_{S(n)}$ converges in probability to $X_* = \liminf X_n$. Integrating $X_{T(n)}$ and $X_{S(n)}$ and using the amart property, we can conclude that $X^* = X_*$. The a.s. convergence of L_1-bounded amarts follows by truncation or stopping: The class of L_1-bounded amarts is closed under both operations [**3**, **21**]. (An even shorter proof of this convergence was given by A. Dvoretzky [**18**].) Convergence of uniformly integrable amarts is already sufficient to derive the Radon-Nikodým theorem. If the martingale property is defined, as usual, using the conditional expectation, an obvious *circulus viciosus* occurs when the martingale theorem is used to obtain the Radon-Nikodým theorem. As a solution, but not the most economical one, it is possible to define the conditional expectation without the Radon-Nikodým property, as in Meyer [**30**, p. 153]. The amart approach seems better.

But what about really new results and applications? High hopes have attended the beginnings of the amart theory, but after ten years we know some of its limitations. Real-valued amarts indexed by positive integers are too close to martingales to have striking, new applications. This follows from the characterization in **[25]**: A sequence (X_n) is an amart if and only if $X_n = Y_n + Z_n$, where (Y_n) is a martingale and (Z_n) is an amart dominated by a positive supermartingale converging to zero. Fortunately, in the Banach-valued case this limiting characterization applies only to "uniform amarts", but not to other classes of amarts. On directed sets, where some of the deepest applications of amarts were obtained, the decomposition $Y_n + Z_n$ holds, but gives less information, because supermartingales in general do not behave as well as amarts.

First we survey Banach-valued amarts. The modern approach to probability in abstract spaces consists in exactly matching the convergence property to the geometry of the space. *Example.* L_1-bounded E-valued martingales converge a.s. if and only if the Banach space E has the Radon-Nikodým property (the Radon-Nikodým theorem holds for E-valued measures; see A. and C. Ionescu Tulcea **[26]**, A. Bellow **[8]**, and S. D. Chatterji **[14]**). This remarkable result gives the only known characterization of a geometric property of Banach spaces in terms of martingales; there are many such characterizations in terms of amarts. The net $E(X_T)$ may converge strongly, weakly, or weak*; the corresponding notions are *strong*, *weak*, or *weak** amart. A notion properly between weak and strong amarts is that of *weak sequential* amart, defined by the property that $E(X_{T(n)})$ converges weakly for every increasing sequence of simple stopping times (T_n). Let E be a Banach space with the Radon-Nikodým property and separable dual. A strong amart of *class* (B), i.e., such tha $\sup(E\|X_T\|\colon T \text{ in } \Sigma) < \infty$, converges weakly a.s. **[13]**. Strong convergence may fail **[13]**; it holds if and only if the Banach space E is finite dimensional (Bellow **[6]**; the proof uses the Dvoretzky-Rogers characterization of finite dimensionality in Banach spaces). A reader interested in strong convergence should look up Bellow's uniform amarts **[7]**, and also a more general, but still *probabilistic*, notion (the optional sampling theorem holds) of *pramart* **[34]**. Pramarts that have an L_1-bounded subsequence converge strongly a.s. in Banach spaces with the Radon-Nikodým property (M. Talagrand **[41]**). Like real amarts, uniform amarts and pramarts converge in the discrete case and have regular paths in the continuous case, the passage from one to the other being accomplished by Doob's method of optional stopping (see B. D. Choi and the reviewer **[15]** and N. Frangos **[24]**). Weak a.s. convergence holds for weak sequential amarts under the same assumptions as for strong amarts, but strong amarts have a more impressive Riesz decomposition: The "potential" part converges to zero in Pettis norm **[22]**. Weak amarts of class (B) converge weakly a.s. if and only if E is reflexive **[10]** (the proof uses H. Rosenthal's theorem that a bounded sequence in a Banach space has a subsequence that is either weakly Cauchy or equivalent to the unit basis in l_1). Also, the separability of the dual is needed for weak a.s. convergence of strong amarts. This difficult result was proved in stages. First W. J. Davis and W. B. Johnson **[16]** gave examples where the dual is not separable and convergence fails; then A. Brunel and the reviewer **[11]** proved that separability of the dual is necessary for the weak convergence of weak sequential amarts; finally, Edgar **[19]** proved the same for strong amarts. The result of **[11]** has a corollary that may be stated without amarts and clearly is a basic result in functional analysis. Say that a sequence of E-valued random variables converges *scalarly* if there is a random variable X such that for each functional f in the dual E^* of E, $f(X_n)$ converges a.s. to $f(X)$. Weak a.s. convergence corresponds, of course, to the case when the exceptional null set does not depend on f. If the dual is separable, then it suffices to consider countably many f's; hence, for bounded sequences, scalar convergence implies weak a.s. convergence. The converse, that if scalar convergence implies weak a.s. convergence then the dual is separable, lies deeper; it is a consequence of [**11**]. A different way to state this result is to say that while the Radon-Nikodým property of the space corresponds to strong a.s. convergence of martingales, the Radon-Nikodým property of both the space and the dual corresponds to weak a.s. convergence of amarts. This formulation is equivalent, by a theorem of Stegall that the dual of a Banach space with the Radon-Nikodým property has this property if and only if it is separable. Finally, for set-valued processes in the dual of a Banach space, the right notion is the weak* amart. It converges only weak* a.s. (S. D. Bagchi [**4**]), but even martingales cannot do any better (J. Neveu [**38**]).

We now discuss amarts indexed by directed sets. The process $(X_t, \mathcal{L}_t\colon t \text{ in } J)$ is now indexed by a set J filtering to the right. In this setting *essential convergence* replaces a.s. convergence, but the entire pathology due to the lack of total order may already exist in the presence of a countable cofinal subset when essential convergence is equivalent to a.s. convergence; therefore we will assume that such a set exists, and consider only a.s. convergence. It is known that L_1-bounded martingales converge a.s. under Krickeberg's Vitali condition V [**27**; and **39**, p. 99]. As recently shown, V is not necessary [**33**], and a weaker condition C is sufficient [**36**] and necessary (Talagrand [**42**]). What about V? This condition admits of a simple probabilistic formulation in terms of stopping times: V holds if and only if for each adapted net of events (A_t), $\limsup A_t$ may be approximated in probability by A_T with T in Σ. A glance at the proof of the amart convergence theorem sketched above will suggest that V may be the condition appropriate for convergence of amarts, and, indeed, K. Astbury [**2**] proved that V holds if and only if all L_1-bounded amarts converge a.s. The connection between V, Σ, and convergence is even more basic: V holds if and only if the convergence in probability of $(X_T\colon T \text{ in } \Sigma)$ implies a.s. convergence of $(X_t\colon t \text{ in } J)$ [**35**]. The situation is typical: To each Vitali condition there correspond an appropriate class of simple stopping times and a class of amarts that converges exactly under this condition. Thus the "ordered" Vitali condition V' reads like V except that the stopping times are *ordered*, by which we mean that the values taken on by each T are totally ordered. V' is sufficient for convergence of L_1-bounded submartingales (Krickeberg [**28**]), but is not necessary [**35**]; it is necessary and sufficient for convergence of L_1-bounded "ordered" amarts [**35**]. There are other examples: e.g., the Vitali conditions V_p, $1 \leqslant p < \infty$, "multivalued stopping times with overlap bounded in L_p," and "multivalued amarts" bounded in L_q $(1/p + 1/q = 1)$ [**34**]. (In this particular instance martingales can do as well as amarts; see Millet [**31**].) The application mentioned above to martingales in the plane consists in the proof that such martingales under conditional independence are amarts with respect to totally ordered filtrations, to which Astbury's theorem applies. Similarly, Edgar's *additive* amarts [**20**] include John Walsh's *strong martingales*, a generalization of sums of independent random variables in the plane, but they also include properly defined strong submartingales (Millet [**32**]) and quasimartingales. Continuous-parameter additive amarts lie still in the future. There are also applications to the derivation of set-functions, a setting in which the Vitali conditions first appeared. Recall that V is satisfied on the line—this is the original theorem of Vitali that gave the condition its name—and also in the plane if the partitions are generated by rectangles such that the ratio of sides remains bounded. Now the derivatives of positive superadditive set-functions are amarts and, therefore, converge under V [**35**]. In imitation of Doob's classical observation that derivatives of *additive* set-functions are martingales, we could prove that derivatives of superadditive set-functions are supermartingales, but this would not be useful, because supermartingales need not converge under V. To be sure, one could also consider the process (X_T: T in Σ), which is a supermartingale, and prove that V' is equivalent to the convergence of all such supermartingales [**35**]; the approach via amarts is more general.

The book under review is composed of two parts. The first part, by Alan Gut, based on lectures delivered in 1979, is a clear and competent exposition of the real-valued theory indexed by N and $-N$, as it appeared to the author at the time. Some proposed applications now seem rather naive. The Marcinkiewicz strong law of large numbers is concerned with processes $X_{-n} = n^{-1}(Y_1 + \cdots + Y_n)^r$, where $1 \leqslant r \leqslant 2$ or $0 \leqslant r \leqslant 1$, the random variables Y_i are independent and identically distributed, and Y_1 is in L_r. In the first case ($1 \leqslant r \leqslant 2$), it is also assumed that $EY_1 = 0$; the amart property is now derived from the Marcinkiewicz theorem, which asserts the convergence of X_{-n}; unfortunately, this approach does not lead to any applications of the amart theory. In the second case ($0 < r \leqslant 1$), if the Y_n's are positive, then X_{-n} is not only an amart, but also a (reversed) submartingale, as shown by Edgar and the reviewer [**23**]; thus, this is not an example of "an amart which is not a martingale, submartingale, etc." (p. 30). However, in two parameters the Marcinkiewicz averages from a positive, reversed "one-submartingale" that converges if $L \operatorname{Log} L$ is bounded, because it is an amart with respect to a totally ordered filtration [**23**]. This proof is not included in the book (nor is any other nontrivial application of amarts). Nevertheless, on balance, the first part of the book could be a suitable exposition of the theory for a reader who did not know much about either amarts or martingales. The same cannot be said about the second part.

The second part of the book, by Klaus D. Schmidt, presents an integrated exposition of amarts indexed by N and $-N$. The term *integrated* is to be taken in its mathematical sense: The integral was applied to sequences of random variables to transform them into sequences of measures or, as the author says, "set function processes". Since simple stopping times take only finitely many values, the integration presented no difficulties. There are no new applications. Via appropriate Radon-Nikodým theorems, the new theory is essentially equivalent to the old one, but simplicity and the probabilistic meaning of stochastic processes and stopping times are lost. In the past the measure-theoretic approach sometimes did advance martingale theory: The article of A. E. Andersen and B. Jessen [**1**] anticipated supermartingales (see Doob [**17**, p. 630]). Set function processes, however, introduce no new hard arguments.

In the Banach-valued case the difficult proofs are omitted, and the results are sometimes misstated. It is not true that "Every weak amart is a uniform amart" (p. 163). Some proofs are incorrectly sketched: On p. 164 it is said that "From an example of Brunel and Sucheston [3] it can be seen that the separability of the dual is a necessary condition in the weak sequential amart convergence theorem." An example, of course, will not do to prove the necessity of a condition. The presentation of H. Heinich's and N. Ghoussoub's "order amarts" in Banach lattices seems better. On the positive side, in the References an effort was made to give a fairly complete list of articles on amarts and related topics: 154 items. *Comments.* A correction to the referenced announcement of U. Krengel was published by Brunel and Krengel [**9**]. Some of the listed articles are wrong. Such is the case of papers by A. Korzeniowski (first reference) and B. D. Choi, as pointed out by this reviewer in *Mathematical Reviews* (see **80a**:60070 and **80g**:60052).

In conclusion, in this book deeper aspects of amarts are either completely omitted (directed sets and continuous parameter) or presented without proof (connection between a.s. convergence and geometry of Banach space). A reader interested in the subject would do well to consult some of the articles listed below rather than the book.

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