#51 Define x_n by $x_n = x_{n-1} + \frac{1}{2}x_{n-2}$, $x_0 = 0$, $x_1 = 1$. Prove that for $n > 8$, x_n is not an integer.
#68 Find, asymptotically, the number of lattice points in the disc $x^2 + y^2 \leqslant R^2$ as $R \to \infty$.
#73 Given n points in the unit square, there is a shortest curve connecting them. Estimate the longest this curve can be.
#82 Show that if $f(x)$ and $f''(x)$ are bounded, then $f'(x)$ is. (Here $f(x) \in C^2$, and the domain is the whole line.)
#90 Can the positive integers be partitioned into at least two arithmetic progressions such that they all have different common differences?
#96 Show that $1 + n/1! + n^2/2! + \cdots + n^n/n! \sim \frac{1}{2}e^n$.
#109 At each plane lattice point there is placed a positive number in such a way that each is the average of its four nearest neighbors. Show that all the numbers are the same!

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Clifford analysis, By. F. Brackx, R. Delanghe, and F. Sommen, Research Notes in Mathematics, Vol. 76, Pitman Advanced Publishing Program, Boston, 1982, 308 pp., $19.95. ISBN 0-2730-8535-2

1. Clifford analysis. What is Clifford analysis? The general answer is that it is the development of a function theory for functions which map $\mathbf{R}^n$ into a universal Clifford algebra with a goal being to generalize to this setting properties of holomorphic functions of one complex variable. Other goals are to relate the monogenic functions, the functions which correspond to holomorphic functions in Clifford analysis, to distributions with values in a Clifford algebra and to study the duals of monogenic functions.

In this first section we define universal Clifford algebra and introduce topological and algebraic structures and spaces of test functions and distributions with values in a certain Clifford algebra; although of a rather technical nature, we need these basic definitions and concepts at our disposal in order to be able to compare the Clifford analysis with previous work and to obtain an understanding of Clifford analysis in its generality as presented in the book under review. In subsequent sections we will discuss motivation for the study of Clifford analysis and topics in the analysis, and we will make some conclusions concerning this book.

Let $V_{n,s}$, $0 \leqslant s \leqslant n$, $n \geqslant 1$, be a real n-dimensional vector space with basis $\{e_1, e_2, \ldots, e_n\}$ and provided with a bilinear form $(a|b)$, $a, b \in V_{n,s}$, such that

$$\begin{aligned} (e_i|e_j) &= 0, \quad && i \neq j, \\ (e_i|e_i) &= 1, && i = 1,\ldots,s, \\ (e_i|e_i) &= -1, && i = s+1,\ldots,n. \end{aligned}$$

Let PN denote the power set of $N = \{1,\ldots,n\}$. Let $C(V_{n,s})$ denote the 2^n-dimensional real vector space with basis

$$\{e_A = e_{\alpha_1}e_{\alpha_2} \cdots e_{\alpha_k} : A = \{\alpha_1, \alpha_2, \ldots, \alpha_k\} \in PN, \\ 1 \leqslant \alpha_1 < \alpha_2 < \cdots < \alpha_k \leqslant n\},$$

where the product is defined by

$$e_A e_B = (-1)^{n((A \cap B)\setminus S)}(-1)^{p(A,B)} e_{A \triangle B}$$

and the distributive law with $S = \{1,\ldots,s\}$, $n(A)$ being the number of elements in $A \in PN$, and

$$p(A, B) = \sum_{j \in B} p(A, j), \qquad p(A, j) = n\{i \in A : i > j\}.$$

Writing $e_\varnothing = e_0$, one can check that e_0 is the multiplicative identity in $C(V_{n,s})$, and we have $e_i e_j = -e_j e_i$, $i \neq j$. In fact $C(V_{n,s})$ is a linear, associative, noncommutative algebra over $\mathbf{R}^1$ and is called the universal Clifford algebra over $V_{n,s}$ after W. K. Clifford [**5**].

$C(V_{1,0})$ is (is isomorphic to) the complex numbers $\mathbf{C}^1$. $C(V_{2,0})$ is the algebra of real quaternions. $C(V_{3,3})$ is the Pauli algebra. $C(V_{4,1})$ is the Dirac algebra.

The construction of Clifford algebras by purely algebraic means can be seen in [**4, 43**, or **39**]. The Clifford algebra under consideration in the book under review is $C(V_{n,0})$ which will be denoted $\mathscr{A}$ in the remainder of this review.

An element $a \in \mathscr{A}$ is called a Clifford number and can be written as

$$a = \sum_A a_A e_A, \qquad A \in PN, a_A \in \mathbf{R}^1.$$

The conjugate (involution) of $a \in \mathscr{A}$ is defined by

$$\bar{a} = \sum_A a_A \bar{e}_A \quad \text{where } \bar{e}_A = (-1)^{n(A)(n(A)+1)/2} e_A.$$

For $a = \Sigma_A a_A e_A$ put $[a]_0 = a_0$, the coefficient of e_0. An inner product on $\mathscr{A}$ can be defined by putting

$$(a, b)_0 = 2^n[a\bar{b}]_0, \qquad a, b \in \mathscr{A},$$

and then a norm on $\mathscr{A}$ can be defined by

$$|a|_0 = ((a, a)_0)^{1/2}.$$

We have that $\mathscr{A}$ is a finite-dimensional real H_*-algebra.

The concept of unitary left (right) $\mathscr{A}$-module is defined exactly as in the algebraic setting. The algebraic structure of the set of all monogenic functions and the algebraic structures of the various spaces of distributions with values in a Clifford algebra are those of unitary $\mathscr{A}$-modules. Any concept that we define for unitary left $\mathscr{A}$-module can be similarly defined for unitary right $\mathscr{A}$-module, and similarly all results obtainable for "left" are also obtainable for "right".

Let $X_{(l)}$ and $Y_{(l)}$ be unitary left $\mathscr{A}$-modules. A mapping T: $X_{(l)} \to Y_{(l)}$ is a left $\mathscr{A}$-linear operator if for all $f, g \in X_{(l)}$ and $a \in \mathscr{A}$, $T(af + g) = aT(f) + T(g)$. If $Y_{(l)} = \mathscr{A}$, T is called a left $\mathscr{A}$-linear functional on $X_{(l)}$ and $T(f) = \langle T, f \rangle$. $X^*_{(l)}$ denotes the set of all bounded left $\mathscr{A}$-linear functionals on $X_{(l)}$ and is a submodule of the unitary right $\mathscr{A}$-module of all left $\mathscr{A}$-linear functionals on $X_{(l)}$. Let Ω be an open subset of $\mathbf{R}^n$. $D(\Omega; \mathscr{A})$ denotes the unitary bi-$\mathscr{A}$-module (i.e. both left and right) of $\mathscr{A}$-valued infinitely differentiable functions in Ω having compact support contained in Ω. Any element $f(x) \in D(\Omega; \mathscr{A})$ has the form $f(x) = \Sigma_A f_A(x) e_A$, where $f_A(x) \in D(\Omega; \mathbf{R}^1)$, the Schwartz test space. The set of all bounded left $\mathscr{A}$-linear functionals on $D_{(l)}(\Omega; \mathscr{A})$ (i.e. $D(\Omega; \mathscr{A})$ considered as a left $\mathscr{A}$-module) is the space $D^*_{(l)}(\Omega; \mathscr{A})$ of left $\mathscr{A}$-distributions in Ω. Similarly, we define $S(\mathbf{R}^n; \mathscr{A})$ and tempered left $\mathscr{A}$-distributions $S^*_{(l)}(\mathbf{R}^n; \mathscr{A})$ corresponding to the Schwartz spaces $\mathscr{S}(\mathbf{R}^n)$ and $\mathscr{S}'(\mathbf{R}^n)$, and $E(\Omega; \mathscr{A})$ and the left $\mathscr{A}$-distributions with compact support in Ω, $E^*_{(l)}(\Omega; \mathscr{A})$, corresponding to the Schwartz spaces $\mathscr{E}(\mathbf{R}^n)$ and $\mathscr{E}'(\mathbf{R}^n)$.

Representation results for the $\mathscr{A}$-linear functionals are important for calculations in the book and are obtained by using the fact that $\mathscr{A}$ is a trace algebra, with the trace τ being given here by

$$\tau(ab) = (a, \bar{b})_0, \qquad a, b \in \mathscr{A}.$$

The duals $X^*_{(l)}$ of certain unitary $\mathscr{A}$-modules $X_{(l)}$ are characterized in terms of real linear functionals; a Hahn-Banach theorem, a Riesz representation theorem, and several other fundamental properties for the Clifford analysis setting are proved by reduction to classical results.

Clifford analysis then is concerned with the study of functions of the form $f(x) = \Sigma_A f_A(x) e_A, f_A$: $\mathbf{R}^n \to \mathbf{R}^1$, which generalize the holomorphic functions of one complex variable, and seeks to develop a function theory for these functions which generalizes the theory of holomorphic functions. The book under review does this for the Clifford algebra $\mathscr{A}$, studies the $\mathscr{A}$-linear functionals defined on these functions, and relates these functions to $\mathscr{A}$-distributions.

2. Motivation for Clifford analysis. The use of the Clifford algebra of real quaternions, $C(V_{2,0})$, as a tool in mathematical physics was greatly diminished after the development of vector calculus, which began about 1880; from this time until approximately 1970 only sporadic activity in both mathematical analysis and physics occurred with respect to research in quaternionic analysis and, more generally, Clifford analysis. Since about 1930 Clifford algebras have been studied in relation to the theory of spinors; see [**4**, **29**, and **44**] for example. Further, Clifford algebras and spinors have been studied extensively in the algebraic sense during this period. Analytically, a flurry of activity was begun in approximately 1930 by Fueter (see §3.1 below) and was continued by his colleagues until the late forties.

In the middle to late sixties quaternionic and Clifford analysis were revived in both mathematical analysis and mathematical physics. The authors of the book under review have been most responsible for the development of a function theory in arbitrary Clifford analysis, and Delanghe [**8**–**11**] has been the leader in this effort. I leave to these authors to state their motivation in studying Clifford analysis.

The motivation for the revival of Clifford analysis, and more specifically quaternionic analysis, in mathematical physics is noted in the very interesting article [**14**] where it is argued that calculations involving complex quaternions —quaternions with complex numbers as coefficients of the basis elements—suggest relativistic generalizations of quantum theory and a new spin-$\frac{1}{2}$ wave equation. Recently complexified quaternionic analysis has been discovered to play a role in problems in electrodynamics and in analyzing four-dimensional σ-models and Yang-Mills fields. (See [**26** and **33**–**35**]. See also [**54**] for spin-$\frac{1}{2}$ massless field results using complexified quaternionic analysis and for related references.)

3. Topics in Clifford analysis. We now present a discussion of the major topics in the book under review with emphasis on the history of the topic and its connections with related subjects.

3.1. *Monogenic functions.* The original analysis of quaternion valued functions of a quaternion variable by Hamilton [**28**] and his followers was developed as a theory of several real variables; they did not consider any special class which corresponded to the holomorphic functions of a complex variable. Such a consideration was not made until the work of Fueter [**19**–**22**] beginning in approximately 1930. Fueter defined a class of quaternionic functions that are analogous to the holomorphic functions, and he chose the Cauchy-Riemann method to do this. He called a quaternion valued function

$$f(t + x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

of the quaternion variable $t + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ left regular (right regular) [**20**, p. 310] if

$$\Gamma f = 0 \quad (f\Gamma = 0) \tag{1}$$

(see (4) below) where Γ is the generalized Cauchy-Riemann operator

$$\Gamma = \frac{\partial}{\partial t} + \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}. \tag{2}$$

Fueter developed a function theory for his regular functions which included analogues of the Caucy theorem, Cauchy integral theorem, Liouville's theorem, and Laurent series. A bibliography of the work of Fueter and his collaborators is contained in [**27**].

In 1973 Deavours published the article [**7**], which had as a primary purpose the introduction of the work of Fueter to readers in the United States. In 1979 Sudbery [**56**], noting the still relative obscurity of the Fueter theory, gave a self-contained account of the basic quaternionic analysis together with some new results. Sudbery showed that the choice of Fueter to define his regular functions as those which satisfy the Cauchy-Riemann condition (1) was the correct choice to obtain a function theory which appropriately generalizes the concept of holomorphic function in complex analysis. A theory of quaternionic power series leads to a class of functions which is too large for its elements to be called regular because such a theory will be the same as a theory of quaternion valued real analytic functions on $\mathbf{R}^4$ [**56**, p. 205]. On the other hand if one defines a quaternionic function to be regular if it possesses a quaternionic derivative in the usual sense, then the class of functions which would be regular consists of only constant and linear functions [**56**, Theorem 1, p. 206], which is too small a class of functions to be comparable with holomorphic functions in complex analysis.

The concept of monogenic function studied in the book under review is a generalization of the holomorphic functions in complex analysis to functions with values in the universal Clifford algebra $\mathscr{A} = C(V_{n,0})$. Let Ω be an open subset of $\mathbf{R}^{m+1}$, where $m \leqslant n$, with n being the dimension of the subspace $V_{n,0}$ of $\mathscr{A}$ having $\{e_1, e_2, \ldots, e_n\}$ as basis. Define the generalized Cauchy-Riemann operator D by

$$D = \sum_{k=0}^{m} e_k \frac{\partial}{\partial x_k}. \tag{3}$$

A function $f \in C^1(\Omega; \mathscr{A})$ is said to be left (right) monogenic in Ω if and only if

$$Df = \sum_{k=0}^{m} \sum_{A} e_k e_A \frac{\partial f_A(x)}{\partial x_k} = 0 \tag{4}$$
$$\left(fD = \sum_{k=0}^{m} \sum_{A} e_A e_k \frac{\partial f_A(x)}{\partial x_k} = 0 \right)$$

in Ω. $M_{(r)}(\Omega; \mathscr{A})$ $(M_{(l)}(\Omega; \mathscr{A}))$ will denote the right (left) $\mathscr{A}$-module of left (right) monogenic functions in Ω. In the case of $\mathscr{A} = C(V_{1,0})$, the complex number system, (4) becomes the Cauchy-Riemann system when $m = n = 1$. We note that the regular function of Fueter is not a special case of the monogenic function; the regular function is a quaternion valued function of a quaternion, a 4-dimensional domain point, while monogenic functions having values in $C(V_{2,0})$ have as domain an open subset of $\mathbf{R}^{m+1}$, with $m + 1$ being at most 3 since $m \leqslant n = 2$ for monogenic functions in this case. Certainly, however, the monogenic functions are motivated by the regular functions, and, as in the case of the regular function, the use of the generalized Cauchy-Riemann system (4), rather than a power series or derivative definition, to define monogenicity is the correct method to obtain a theory which corresponds to that of the holomorphic functions in complex analysis. In this book the authors develop a general function theory for monogenic functions. Clifford analytic versions of Cauchy's theorem, Cauchy's integral formula, maximum modulus theorem, Morera's theorem, Taylor series, Laurent series, meromorphic function, Mittag-Leffler theorem, Liouville theorem, residue theory, and Cauchy residue theorem are obtained.

A function theory for functions with values in an arbitrary Clifford algebra had its beginnings in the early forties [**1, 42**]. But a systematic study of Clifford analysis was not begun until the middle to late sixties; the papers [**8–11**, **30**, and **32**] must be mentioned in this regard. The four papers of Delanghe [**8–11**] lay the foundation for the basic properties of the monogenic functions presented in the book under review.

3.2. *Partial differential equations.* Recall that the equation $Df = 0$ given in (4) reduces to the classical Cauchy-Riemann system for holomorphic functions of one complex variable in the case $m = n = 1$. Similarly, the equation $Df = 0$ of (4) is equivalent to a linear system of 2^n homogeneous partial differential equations of the first order with constant coefficients for $f \in C^1(\Omega; \mathscr{A})$, $\mathscr{A} = C(V_{n,0})$.

In [**13**] Douglis studied elliptic systems of partial differential equations that can be decomposed into canonical subsystems of $2r$ equations with $2r$ unknowns and two independent variables [**13**, pp. 260–261]; he called such canonical systems generalized Beltrami systems if the equations are homogeneous and contain no terms of order zero in the dependent variables. Douglis showed with the aid of a commutative associative algebra that the generalized Beltrami system could be written in a shortened form noted in (5) below. The algebra used was a commutative associative algebra over the reals generated by two elements i and e subject to the multiplication rules

$$i^2 = -1, \quad ie = ei, \quad e^r = 0.$$

The elements of this algebra are the linear combinations with real coefficients of the $2r$ linearly independent elements e^k, ie^k, $k = 0, 1, \ldots, r-1$, where $e^0 = 1$. An element of the algebra can be written $\sum_{k=0}^{r-1} c_k e^k$, where c_k, $k = 0, 1, \ldots, r-1$, is a complex number, and the element is called a hypercomplex number. A hypercomplex function is a function from the (x, y) plane into this algebra and has the form

$$f(x, y) = \sum_{k=0}^{r-1} f_k(x, y) e^k,$$

where each $f_k(x, y)$, $k = 0, 1, \ldots, r-1$, is complex valued. The generalized Beltrami system of Douglis can now be written

$$\Lambda f = 0 \tag{5}$$

where Λ is the differential operator

$$\Lambda = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} + ea\frac{\partial}{\partial x} + eb\frac{\partial}{\partial y}$$

with a and b being coefficients in the system. A continuously differentiable hypercomplex function $f(x, y)$ which satisfies (5) is called a hyperanalytic function, and Douglis developed a function theory for the hyperanalytic functions. (We note that [**13**] contains several references to other publications which involve the algebra used by Douglis in studies of elliptic equations. Also given are references which concern more general commutative algebras in association with generalized function theory and which concern the study of partial differential equations associated with the work of Fueter and in which noncommutative algebras are used.)

Gilbert and Hile ([**24, 25**], and further papers), building upon the work of Douglis, have considered more general systems, which take the shortened form of a hypercomplex equation with the use of the commutative associative algebra of Douglis, and solutions of the hypercomplex equation are called generalized hyperanalytic functions. Gilbert and Hile developed a function theory for the generalized hyperanalytic functions.

Of course the algebra used by Douglis and Gilbert and Hile is not a universal Clifford algebra since it is commutative. But the point is that the study of systems of partial differential equations with solutions having values in an algebra has existed for quite some time and is in the spirit of the book under review with a function theory being developed for the hyperanalytic functions.

In addition to developing a function theory for the left (right) monogenic functions, the authors of *Clifford analysis* also study the equation $Df = g$, where g is an element of a given class of functions or distributions and D is the differential operator of (3). In particular if $g \in E_{(r)}(\Omega; \mathscr{A})$ $(\in D^*_{(l)}(\Omega; \mathscr{A})$, $\in S^*_{(l)}(\mathbf{R}^{m+1}; \mathscr{A}))$, Ω being an open subset of $\mathbf{R}^{m+1}$, the authors prove that there exists $f \in E_{(r)}(\Omega; \mathscr{A})$ $(\in D^*_{(l)}(\Omega; \mathscr{A})$, $\in S^*_{(l)}(\mathbf{R}^{m+1}; \mathscr{A}))$ such that $Df = g$. The proofs use partition of unity techniques in the setting of Clifford analysis. These results are an extension to Clifford analysis of classical results concerning the solution of partial differential equations with distribution solutions. The first investigations of this type were undertaken at approximately the same time by Ehrenpreis [**15–18**] and Malgrange [**40**] who proved the existence of solutions to the equation $PT = S$ [**15**, p. 883], for S in various spaces of functions or distributions, including the Schwartz distributions $\mathscr{D}'$, and P a partial differential operator. Both Ehrenpreis and Malgrange extended some of their results to systems of partial differential equations. Hörmander [**31**] showed that $PT = S$ had a solution $T \in \mathscr{S}'$ if $S \in \mathscr{S}'$. Investigations of this type continue corresponding to more general partial differential operators. (On p. 196 of the book under review the references [**E2**] and [**E3**] are incorrect. The second paper in this series of papers by Ehrenpreis, of which there are four, is omitted. The correct references to this series of papers are given in the bibliography of this review in [**15–18**].)

3.3. *Duality in function theory. Analytic functionals.* Recall that $M_{(r)}(\Omega; \mathscr{A})$ is the right $\mathscr{A}$-module of all left monogenic functions which map $\Omega \subset \mathbf{R}^{m+1}$ into $\mathscr{A} = C(V_{n,0})$, $m \leqslant n$. A problem considered in the book is to characterize the dual $M^*_{(r)}(\Omega; \mathscr{A})$, the set of all bounded right $\mathscr{A}$-linear functionals on $M_{(r)}(\Omega; \mathscr{A})$, and the bidual of $M_{(r)}(\Omega; \mathscr{A})$. The results are that $M^*_{(r)b}(\Omega; \mathscr{A})$, the set $M^*_{(r)}(\Omega; \mathscr{A})$ provided with the strong topology, is topologically isomorphic to the set $\tilde{M}_{(l)}(\mathbf{R}^{m+1} \setminus \Omega; \mathscr{A})_{\text{ind}}$, a certain left $\mathscr{A}$-module of right monogenic functions in $\mathbf{R}^{m+1} \setminus \Omega$ which is endowed with the inductive limit topology. The isomorphism is defined with the aid of an extension of the notion of the indicatrix of Fantappiè to the Clifford analysis setting. Because $M^*_{(r)b}(\Omega; \mathscr{A})$ is topologically isomorphic to $\tilde{M}_{(l)}(\mathbf{R}^{m+1} \setminus \Omega; \mathscr{A})_{\text{ind}}$, the set $M^*_{(r)}(\Omega; \mathscr{A})$ is called the space of analytic functionals in Ω in analogy with the classical situation which we mention in the next paragraph. Further, the strong dual of $\tilde{M}_{(l)}(\mathbf{R}^{m+1} \setminus \Omega; \mathscr{A})_{\text{ind}}$ is topologically isomorphic to $M_{(r)}(\Omega; \mathscr{A})$;

hence $M_{(r)}(\Omega; \mathscr{A})$ is reflexive. The proofs of the duality results rely upon Runge type theorems for monogenic functions.

The results are completely analogous to the historical situation for duality in function theory. An extensive summary of the work in duality in function theory, begun by Fantappiè and continued by Sebastião e Silva, Köthe, Grothendieck, Tillmann and others, is contained in [36] along with an extensive reference list concerning this and related analysis.

3.4. *Distributional boundary values and integral transforms.* The classical distributional boundary value problem can be described as follows. Given a distribution (generalized function) U on some set, say $\mathbf{R}^1$ or $\mathbf{R}^n$ embedded in $\mathbf{C}^1$ or $\mathbf{C}^n$, construct holomorphic functions which have U as boundary value in the distribution topology as the imaginary part (real part) of the complex variable approaches zero and specify the growth of the holomorphic functions which do this. Conversely, given holomorphic functions with the specified growth prove the existence of a distributional boundary value and recover the holomorphic function from the boundary value. Further, show, if possible, that the boundary value mapping becomes a topological isomorphism between the space of distributions and a space of holomorphic functions defined by the specified growth.

As an example of this problem let us consider the original results of Tillmann [59, p. 110] for the tempered distributions $\mathscr{S}' \equiv \mathscr{S}'(\mathbf{R}^n)$ corresponding to tubes in $\mathbf{C}^n$ defined by the quadrants in $\mathbf{R}^n$. Let $\sigma = (\sigma_1, \ldots, \sigma_n)$ denote any of the 2^n n-tuples whose components are 0 or 1. Consider the growth

$$(6) \quad |f(z)| \leqslant M \prod_{j=1}^{n} \left(1 + |z_j|^2\right)^{m_j} |y_j|^{-1/2-k_j}, \qquad z = x + iy \in \left(\mathbf{C}^1 \setminus \mathbf{R}^1\right)^n,$$

where M is a constant, $(m_1, \ldots, m_n)$ is an n-tuple of nonnegative real numbers, and $(k_1, \ldots, k_n)$ is an n-tuple of nonegative integers. We define the boundary value mapping of $f(z)$ by

$$(7) \qquad \mathrm{BV}(f(z)) = \lim_{\varepsilon \to 0+} \sum_{\sigma} (-1)^{|\sigma|} f(x + i\varepsilon y_\sigma),$$

where $|\sigma| = \sigma_1 + \cdots + \sigma_n$ and $y_\sigma = ((-1)^{\sigma_1}, \ldots, (-1)^{\sigma_n})$. Tillmann proved that if $f(z)$ is holomorphic and satisfies (6) in $(\mathbf{C}^1 \setminus \mathbf{R}^1)^n$, then $\mathrm{BV}(f(z))$ exists in the strong topology of $\mathscr{S}'$ as an element of $\mathscr{S}'$; conversely, any $U \in \mathscr{S}'$ is the boundary value of a holomorphic function which satisfies (6) in $(\mathbf{C}^1 \setminus \mathbf{R}^1)^n$. The set of pseudo-polynomials [59, p. 112] is the kernel of $\mathrm{BV}(f(z))$. $\mathscr{S}'$ is algebraically isomorphic to the quotient space of holomorphic functions in $(\mathbf{C}^1 \setminus \mathbf{R}^1)^n$ satisfying (6) with the pseudo-polynomials as kernel under the boundary value mapping $\mathrm{BV}(f(z))$; and the isomorphism subsequently was proved to be topological when the holomorphic functions are endowed with a suitable topology. Further, the holomorphic function $f(z)$ can be recovered from its boundary value $\mathrm{BV}(f(z))$ by a certain Cauchy integral of $\mathrm{BV}(f(z))$; and the restriction of $f(z)$ to each of the tubes defined by the 2^n quadrants $C_\sigma = \{y \in \mathbf{R}^n\colon (-1)^{\sigma_j} y_j > 0,\ j = 1, \ldots, n\}$ can be recovered by the Fourier-Laplace transform of components of the inverse Fourier transform of the boundary value which have support in the dual cone $\overline{C}_\sigma$ of C_σ.

The original paper on representing distributions as boundary values of holomorphic functions was by Köthe [37] who proved that a generalized function defined on a closed curve in the extended complex plane has a representation as a limit similar to (7) of two functions, one holomorphic in the interior of the domain bounded by the curve and the other holomorphic in the exterior. Tillmann [57] generalized Köthe's theory to unbounded domains in $\mathbf{C}^1$ and to functions of several variables defined on regions in $\mathbf{C}^n$ that are the product of unbounded domains in the plane; in particular Tillmann's theorems apply to half planes in $\mathbf{C}^1$ and to generalized half planes in $\mathbf{C}^n$ defined by the quadrants C_σ in $\mathbf{R}^n$. In [57] Tillmann obtained a characterization of the holomorphic functions which represent the distributions of compact support $\mathscr{E}' \equiv \mathscr{E}'(\mathbf{R}^n)$; the Cauchy integral of $\mathscr{E}'$ elements is used in this analysis. Subsequently, Tillmann considered the $\mathscr{D}'_{L^p} \equiv \mathscr{D}'_{L^p}(\mathbf{R}^n)$ distributions in [58] and the $\mathscr{S}'$ distributions in [59]. In the boundary value problem for $\mathscr{D}'_{L^p}$, the Cauchy integral of such elements is used.

The distributional boundary value process has important physical applications. In quantum field theory and other applications in physics the holomorphic functions need to be defined in tubes in $\mathbf{C}^n$ defined by quadrants, light cones, or more generally open connected cones in $\mathbf{R}^n$ in addition to half planes in $\mathbf{C}^1$. The distributional boundary value U in the tube setting is interpreted to be the vacuum expectation value in a field theory. U is the Fourier transform of a distribution V with support in the dual cone of the cone that defines the tube, and the holomorphic function can be recovered as the Fourier-Laplace transform of V. For functions in tubes defined by a light cone and applications in quantum field theory, see [55] and [6] and the references given there; also see [49] for results and an extensive reference list.

For functions in tube domains in $\mathbf{C}^n$ defined by arbitrary open connected cones in $\mathbf{R}^n$ the most important and original distributional boundary value results have been obtained by V. S. Vladimirov in [61 and 62] and in his papers referenced there.

In addition to the Cauchy and Fourier-Laplace integrals, the Poisson integral of a distribution can be used in certain cases to recover the holomorphic functions from their distributional boundary values even in the most general tube domain setting.

To this point in this section we have concerned ourselves with holomorphic functions and generalized functions that are complex valued. Many of the results surveyed have been extended to vector valued distributions and holomorphic functions, distributions and functions with values in a locally convex topological vector space for example. The first such results were obtained by Tillmann in [60]. An excellent survey of the vector valued results obtained up to 1977 is contained in [41]. Included in this survey are comments concerning ultradistributions as well as distributions; and the survey includes work of Komatsu, Konder, Körner, Martineau, Meise, Petzsche, Roumieu, Vogt, and others associated with this topic. Additionally we must note Sato and other excellent authors [47, 38] in the development of hyperfunctions, a concept which utilizes the boundary values of holomorphic functions in the generalized function sense.

Now let $\vec{\Omega}$ be an open set in $\mathbf{R}^m$ and $\vec{x} \in \vec{\Omega}$. Let $\Omega \subset \mathbf{R}^{m+1}$ be a certain neighborhood of $\vec{\Omega}$. In the book under review boundary values of monogenic functions are obtained in the sense that the existence of the limits

$$\mathrm{BV}^+(f) = \lim_{x_0 \to 0+} f(\vec{x} + x_0), \qquad \mathrm{BV}^-(f) = \lim_{x_0 \to 0+} f(\vec{x} - x_0),$$
$$\mathrm{BV}(f) = \mathrm{BV}^+(f) - \mathrm{BV}^-(f)$$

are studied. Here $(\vec{x} + x_0) \in \Omega$ and the boundary values are obtained on $\vec{\Omega}$. Boundary value results for monogenic functions are obtained in which elements of $D^*_{(l)}(\vec{\Omega}; \mathscr{A})$, $S^*_{(l)}(\vec{\Omega}; \mathscr{A})$, and $E^*_{(l)}(\mathbf{R}^m; \mathscr{A})$ are the boundary values. In the case of $S^*_{(l)}(\vec{\Omega}; \mathscr{A})$ the monogenic functions which obtain these boundary values are characterized by a growth in the Clifford norm that resembles the growth (6) of Tillmann in the scalar valued case. The proof of the existence of boundary values of monogenic functions in $S^*_{(l)}(\Omega; \mathscr{A})$ closely resembles that of Tillmann for the scalar valued case. If $U \in E^*_{(l)}(\mathbf{R}^m; \mathscr{A})$ a Cauchy integral $C(U; x)$, $x = \vec{x} + x_0$, is constructed such that

$$\langle U, \phi \rangle = \lim_{x_0 \to 0+} \int_{\mathbf{R}^m} \phi(\vec{x})(C(U; \vec{x} + x_0) - C(U; \vec{x} - x_0))\, d\vec{x},$$
$$\phi \in D_{(l)}(\mathbf{R}^m; \mathscr{A}),$$

with $C(U; x)$ being left monogenic and satisfying a growth condition in the Clifford norm which characterizes those monogenic functions having $E^*_{(l)}(\mathbf{R}^m; \mathscr{A})$ boundary values in the topology of $D^*_{(l)s}(\mathbf{R}^m; \mathscr{A})$; these results closely resemble the corresponding scalar valued case for generalized functions.

The reader probably is familiar with L. Schwartz's proof that the Fourier transform is a topological isomorphism of $\mathscr{S} \equiv \mathscr{S}(\mathbf{R}^n)$ onto itself [**48**, p. 249], with the same being true for $\mathscr{S}' \equiv \mathscr{S}'(\mathbf{R}^n)$, where the Fourier transform $U = F[V]$ of an element $V \in \mathscr{S}'$ is an element of $\mathscr{S}'$ defined by the Parseval relation [**48**, p. 250]

$$\langle U, \phi \rangle = \langle V, F[\phi] \rangle, \qquad \phi \in \mathscr{S}.$$

A Fourier transform is defined on $S_{(l)}(\mathbf{R}^m; \mathscr{A})$ ($S_{(r)}(\mathbf{R}^m; \mathscr{A})$) and on $S^*_{(l)}(\mathbf{R}^m; \mathscr{A})$ ($S^*_{(r)}(\mathbf{R}^m; \mathscr{A})$) in the function and $\mathscr{A}$-linear functional sense, respectively, both of which exactly parallel the scalar valued case. The same is true of Clifford spaces corresponding to the spaces $\mathscr{D}$, $\mathscr{Z}$, $\mathscr{D}'$ and $\mathscr{Z}'$ and corresponding to the results of Ehrenpreis and Gel'fand and Shilov regarding the Fourier transform relation of these spaces. In fact it is intriguing how closely the Clifford analysis results related to the Fourier transform in $S^*_{(l)}(\mathbf{R}^m; \mathscr{A})$ and in $D^*_{(r)}(\mathbf{R}^m; \mathscr{A})$ resemble the classical situation. As another example, the $S^*_{(l)}(\mathbf{R}^m; \mathscr{A})$ Fourier transform of an element in $E^*_{(l)}(\mathbf{R}^m; \mathscr{A})$ is a Clifford analytic Fourier-Laplace integral which satisfies the classical growth in Clifford norm; this together with the converse yield the Paley-Wiener-Schwartz theorem in the Clifford analysis setting. As still another example, the boundary value of the Clifford analytic Laplace (Fourier-Laplace) transform of an element $V \in S^*_{(l)}(\mathbf{R}^m; \mathscr{A})$ having various support properties is the

$S^*_{(l)}(\mathbf{R}^m; \mathscr{A})$ Fourier transform of V, and this Laplace transform is a monogenic function which satisfies a growth in Clifford norm similar to the growth in the scalar valued case. Conversely, a monogenic function satisfying this growth has as boundary value the Fourier transform of some $V \in S^*_{(l)}(\mathbf{R}^m; \mathscr{A})$ and the Laplace transform of V has this same boundary value.

4. Conclusions. Topics included in the book in addition to those discussed in §3 are the construction of Hilbert modules with reproducing kernels and extension of the classical HL_2 and H^2 spaces and their corresponding Bergman and Szegö kernels to the setting of monogenic functions, analytic functions and functionals on the unit sphere, L^2 functions on the unit sphere, a generalized Fourier-Borel transform, and a generalized Radon transform.

As noted in §2 a revival of quaternionic and Clifford analysis occurred in the late sixties, with the authors of this book being the leaders in the study of Clifford analysis. This book is the only one published to this point which surveys the work in Clifford analysis since 1970. In the past year several important papers in this subject have appeared which give evidence of the continuing interest in the subject; these are [**45, 46**, and **51–54**]. In addition, several papers related to Clifford analysis appear in [**23**]. Other authors are beginning to be interested in Clifford analysis in their research; for example we reference Zayed [**63**] who presents Hardy space results for functions with values in a Clifford algebra. Because of the renewed interest in Clifford analysis in both mathematical analysis and mathematical physics, a broad base has been established for continued research in this area. The publication of the book under review is thus especially appropriate and timely now; it will become a fundamental reference for anyone pursuing this subject.

As noted in this review the Clifford analysis contained in this book is with respect to the real finite-dimensional Clifford algebra $\mathscr{A}$. The complexification of Clifford analysis in the general setting has begun in [**46**]; here the Clifford analysis is extended to an analysis over complex finite-dimensional Clifford algebras (i.e. the scalar coefficients in the Clifford numbers are complex numbers instead of just real numbers). As previously noted, complexified quaternionic analysis has been developed and has proved useful in mathematical physics [**26, 33–35, 54**]. The further development of complexified Clifford analysis will certainly be undertaken in the future. Clifford analysis will continue to be used in analysis of systems of partial differential equations, and classical topics concerning complex analysis will be extended to the Clifford analysis setting as in [**63**].

The book under review has several nice structural features. A list of symbols is given at the beginning of the book with the section number where the symbol is introduced; as I read the book I continually found myself looking back at this list to recall symbols. At the end of each chapter the authors have written a section of notes concerning the chapter and a bibliography of source material and related work; in the notes the authors have included historical comments concerning the material of that chapter as well as comments concerning the structure of the analysis in the chapter in some cases. These notes are especially helpful in understanding the development of the subject.

An additional bibliographic list is included at the end of the book as are a complete author index and a word and phrase index.

The book contains some errors, but these are more typographical in nature than mathematical. We do note that on p. 14, line 2↓, $(T\mu)(f) = T(f)\mu$, $f \in X_{(l)}$, $T \in L(X_{(l)}, Y_{(l)})$, $\mu \in \mathscr{A}$, is not well defined since $Y_{(l)}$ is a unitary left $\mathscr{A}$-module; on p. 19, line 3↑, $2^{n/2}|\langle T, f\rangle_A|_0$ should be $2^{n/2}|\langle T, f\rangle_A e_A|_0$; on p. 297, line 18↓, "Laplace" should be "Fourier". The book has been published by photolithographic reproduction of the typed manuscript, and a number of symbols have been typed incorrectly or completely omitted in the manuscript. The omissions include absolute value symbols, inequalities, subscripts, and Greek and script letters. In Chapter 1 $\mathscr{A}$ is typed rather than A in several places; on p. 34, last line, $\mathscr{A}$ is missing; on p. 38, line 7↑, the subscript 1 is missing; on p. 93, line 6↓, $P_k(x)$ should be $P_k f(x)$; on p. 212, last line, α is missing; on p. 256, line 13↓, $S_{(l)}$ should be $S^*_{(l)}$; on p. 261, line 7↓, $\leqslant$ is missing; on p. 280, line 10↓, the upper limit ∞ is missing on the first summation. These are some of the approximately 40 omissions and incorrectly typed symbols that I saw in the book. However, none of the errors or omissions in the typing impede the reading of the mathematics; in every case the authors' intention is easy to see.

Clifford analysis is an excellent summary of the research conducted mainly by the authors over the past 16 years. It is clearly written. It is a timely publication because of the renewal of interest in quaternionic and Clifford analysis in mathematical analysis and mathematical physics and will become an indispensable source for researchers in the subject.

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Note: Two new papers in Clifford analysis have just appeared in Complex Variables Theory Appl. **2** (1983). R. Delanghe is a member of the editorial board of this new journal.

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Cohomology of groups, by Kenneth S. Brown, Graduate Texts in Mathematics, Vol. 87, Springer-Verlag, New York, 1982, x + 306 pp., $28.00. ISBN 0-3879-0688-6

The cohomology theory of abstract groups is a tool kit, in much the same way as is representation theory. One of its attractions is its breadth: the