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INVARIANTS OF FORMAL GROUP LAW ACTIONS[1]

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0. Introduction. In this note, k denotes a field of characteristic $p > 0$, and the letters T, X and Y are formal indeterminants. Let $F\colon k[[T]] \to k[[X,Y]]$ be a (fixed) one-dimensional formal group law [**Dieudonné, Hazewinkel, Lazard, Lubin**] of height $h \geq 0$. Let V denote a $k[[T]]$ module of finite length. Suppose $\operatorname{Ann}(V) = (T^n)$. Let $q = p^e$ denote the least power of p such that $n \leq q$. It follows that the symmetric powers $S_r(V)$ over k become $k[[T]]$-modules, annihilated by T^q, through the formal group law, viz: If $F(T) = X + Y + \sum_{i,j\geq 1} C_{ij}X^iY^j$ and f is in $S_t(V)$ and g is in $S_s(V)$, then

$$T(fg) = fTg + (Tf)g + \sum C_{ij}(T^i f)(T^j g)$$

in $S_{t+s}(V)$.

Denote by $S.(V)$ the symmetric algebraic on V; so

$$S.(V) := \bigoplus_{r>0} S_r(V).$$

Then $S.(V)$ is a $k[[T]]$-module annihilated by T^q. The main purpose of this note is to announce and outline a proof of the theorem below. Several consequences and examples are included.

THEOREM. *Let $S.(V)^F := \{f \in S.(V) : Tf = 0\}$. The set $S.(V)^F$ is a normal noetherian subring of $S.(V)$ of the same Krull dimension. Furthermore, $S.(V)^F$ is factorial.*

1. An outline of the proof. To prove the Theorem one can consider two cases: $\operatorname{ht} F = h = 1$ and $\operatorname{ht} F = h \neq 1$. In case $\operatorname{ht} F = 1$, the action on $S.(V)$ is equivalent to an action of the cyclic group $\mathbf{Z}/qZ$ on $S.(V)$. This case is considered, in full generality, in [**Fossum, Griffith**] and [**Almkvist, Fossum**].

So consider the case $\operatorname{ht} F \neq 1$. It can be shown that there is a fixed power s of p, depending only on $\operatorname{ht} F$, such that $S.(V)^s \subset S.(V)^F$. Then one can extend the action of T to the field of fractions L of $S.(V)$ via

$$T(f/g) = T(fg^{s-1})/g^s.$$

Then one concludes that L^F is a field and

$$S.(V)^F = L^F \cap S.(V),$$

which shows that $S.(V)^F$ is a Krull domain and $S.(V)^F \supset k[S.(V)^s]$, which shows that $S.(V)^F$ is noetherian and $S.(V)$ is integral over $S.(V)^F$. Hence,

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the extension $S.(V)^F \to S.(V)$ is (PDE) (cf. [**Fossum**]). It remains to prove that $S.(V)^F$ is factorial. As $S.(V)$ and $S.(V)^F$ are graded, it is sufficient to consider homogeneous ideals and to show that each homogeneous, prime, divisorial ideal $\mathfrak{p}$ in $S.(V)^F$ is principal. One could accomplish this by using results in [**Waterhouse**], but there is also a straightforward calculation.

Suppose $\mathfrak{p}$ is a homogeneous prime divisorial ideal in $S.(V)^F$. Let $(\mathfrak{p}S.(V))^{**}$ denote the divisorial ideal it generates in $S.(V)$. This ideal is principal, generated by a homogeneous element f in $S.(V)$. It follows that

$$f^{-1}S.(V) = \{x \in L\colon x\mathfrak{p} \subset S.(V)\}.$$

Since $f^{-1}\mathfrak{p} \subset S.(V)$ and $T(\mathfrak{p}) = 0$, it follows easily that $T(f^{-1})\mathfrak{p} \subset S.(V)$. Hence, there is an element $a \in S.(V)$ such that

$$T(f^{-1}) = f^{-1}a, \quad \text{or} \quad T(f^{s-1}) = f^{s-1}a.$$

Since degree $T(f^{s-1})$ = degree f^{s-1}, it follows that degree $a = 0$, or that $a \in k$. Hence $T(a) = 0$. Then

$$T^r(f^{s-1}) = f^{s-1}a^r.$$

Since $T^q = 0$, one obtains that $a^q = 0$ or $a = 0$. Hence, $T(f^{s-1}) = 0$. Then

$$0 = T(f^s) = fT(f^{s-1}) + T(f)f^{s-1} = T(f)f^{s-1}.$$

Hence, $T(f) = 0$, so $f \in S.(V)^F$. From this one concludes that $\mathfrak{p} = fS.(V)^F$. This concludes the (outline) of the proof. Details that are omitted will appear elsewhere.

2. Examples. For each $n \in \mathbf{N}$, set $V_n := k[[T]]/(T^n)$. The Hilbert series of $S.(V_n)^F$ is, by definition, the power series

$$\sum_{r=0}^{\infty} (\mathrm{rk}_k\, S_r(V_n)^F)t^r \in \mathbf{Z}[[t]].$$

Denote this series by $H_t(V_n, F)$. (The dependence on p is implicit.) In case $F(T) = X + Y + XY$ (and so ht $F = 1$) these Hilbert series have been studied extensively in [**Almkvist, Fossum**]. In particular, for $n = q = p^e$,

$$H_t(V_{p^e}, F) = p^{-e}\left\{(1-t)^{-p^e} + \sum_{j=1}^{e}(p^j - p^{j-1})(1-t^{p^j})^{-p^{e-j}}\right\}.$$

Some results are now available for the additive formal group law $F(T) = X+Y$ (so ht $F = 0$). In this case, for $q = p^e$,

$$\begin{aligned} H_t(V_q, X+Y) &= q^{-1}\{(1-t)^{-q} + (q-1)(1-t^p)^{-q/p}\} \\ &= (1-t^p)^{-q/p} + q^{-1}\{(1-t)^{-q} - (1-t^p)^{-q/p}\}. \end{aligned}$$

This can be used to show that

$$H_t(V_{q-1}, X+Y) = (1-t^p)^{-q/p} + (1-t)q^{-1}\{(1-t)^{-q} - (1-t^p)^{-q/p}\}.$$

This rational function of t is not unimodal for most q, and hence $S.(V_{q-1})^{(X+Y)}$ is *not* Gorenstein for these q. Thus $S.(V_{q-1})^{(X+Y)}$ is not Cohen-Macaulay. Evidence suggests that $S.(V_n)^F$ is not Cohen-Macaulay for $n \geq 4$, except in case $p = 2$ and $n = 4$ (cf. [**Stanley**]). The case $p = 2$ has been studied by [**Bertin**]. The Hilbert series

$$H_t(V_4, X+Y+XY) = \frac{1-t+t^2+t^3}{(1-t)^2(1-t^2)^2(1+t^2)},$$

which shows that $S.(V_4)^{X+Y+XY}$ is not Cohen-Macaulay, since it is factorial and not Gorenstein.

The Hilbert series, $p = 2$, for V_4 is

$$H_t(V_4, X+Y) = \frac{(1+t^3)}{(1-t)(1-t^2)^3}.$$

A calculation shows, for $S.(V_4) = k[X_3, X_2, X_1, X_0]$ with $TX_3 = X_2$, $TX_2 = X_1$, $TX_1 = X_0$, $TX_0 = 0$, and $T(fg) = f(Tg) + (Tf)g$, that

$$S.(V_4)^{X+Y} = k[X_0, X_1^2, X_2^2, X_3^2, X_3X_0^2 + X_2X_1X_0 + X_1^3],$$

which is a complete intersection.

Further results will appear in more detail.

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