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# "LA CROIX DES MATHÉMATICIENS": THE EUCLIDEAN THEORY OF IRRATIONAL LINES

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For the modern reader Euclid's Tenth Book is by far the most intimidating portion of the *Elements*, by virtue of its enormous length and the obscurity of its techniques and motives. To approach this material, one requires a key, of the sort the Flemish mathematician-engineer Simon Stevin boasted to possess almost four centuries ago:

> After we had viewed and reviewed the Tenth Book of Euclid treating of incommensurable magnitudes, and also had read and reread several commentators on the same, of whom some judged it for the most profound and incomprehensible matter of mathematics, others that these are most obscure propositions and the cross of mathematicians, and beyond this I persuaded myself (what folly doesn't opinion cause men to commit?) to understand this matter through its causes, and that there are in it none of the difficulties such as one commonly supposes, I have taken it upon myself to describe this treatise.[1]

Stevin's ploy, by which "this whole affair is easy and without difficulty", involved the expression of Euclid's propositions via a calculus of surd quantities, and more recent commentaries, such as those by Heath and by Junge, follow suit in the application of algebraic modes for explaining this material.[2] But for the historically minded reader the issue of interpretation has been complicated by this, for the originators of this theory cannot have had such algebraic modes at hand in their formulation. The project of elucidating the motives underlying Euclid's geometric form of the theory has largely eluded even the best of the modern accounts.[3]

I here propose to offer a view of the geometric problems on which the structure of Euclid's theory is built. This view fills out the details of a sketch I presented in my study of the pre-Euclidean geometry a few years ago and supplements the handy treatment given before that by B. L. van der Waerden.[4] I will show how the essential idea of the theory emerges through consideration

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of the Euclidean constructions of regular plane and solid figures, next trace out the development of this idea in the theory of Book X, and close with some thoughts on the nature of the formal project embraced in this book. It will first be necessary to introduce a few basic notions drawn from the early phases of the study of incommensurable magnitudes.

**I. First steps.** Just when and how the ancient Greeks first discovered the existence of incommensurable magnitudes is a vexed question. On the authority of very late legendizing writers one could assign this discovery to Pythagoreans early in the fifth century B. C., or even to Pythagoras himself a bit earlier still.[5] But evidence drawn from the Presocratics discourages a dating before the middle of the fifth century. Although the context of studies of the regular pentagon and the division of lines according to the "golden section" has been proposed as the initial context of the recognition of incommensurable lines, fourth-century witnesses, like Plato and Aristotle, treat the side and diameter of the square as paradigmatic of incommensurables, while a geometric form of the well-known proof of their incommensurability, founded on the distinction between odd and even integers, offers a perfectly feasible manner for the discovery and first demonstration of this result.[6] Indeed, the special computational difficulties which arise in the evaluation of $\sqrt{2}$ already had by then a history of over a thousand years.[7] In perceiving that such a quantity was in principle inexpressible as a ratio of integers, the Greek geometers would be injecting a characteristically theoretical element into the investigation of a technical matter.

By around 400 B.C. other constructions were recognized to give rise to incommensurable lines through efforts by Theodorus of Cyrene and Archytas of Tarentum, and the general result incorporating these was known, and perhaps first enunciated by Theaetetus of Athens early in the fourth century. Plato gives a loose statement of it in the dialogue named after this mathematician:

> Such lines as square the equilateral and plane number we defined as 'length', but such as square the oblong number as 'powers', for these are not commensurable with the former in respect of length, but rather in respect of the planes which they produce. And concerning the solids another such thing holds.[8]

The sharp arithmetical cast of this formulation is tempered in the statement of the more general condition within the Euclidean theory:

> The squares on lines commensurable in length have to each other the ratio which a square number has to a square number; and [conversely], squares having to each other the ratio which a square number has to a square number will also have the sides commensurable in length...[9]

In this extension, the condition covers not only the case where one of the lines or squares is an integral multiple of the other, but also when the ratio of the one to the other is that of an integer to an integer. Plato's aside on the solid case is of interest, since it makes clear that the condition for the commensurability of lines formed as cubic roots of integral or rational terms was well understood. As the latter has no bearing on the constructions in the Euclidean theory, it does not receive its statement there. But the principles by which Euclid effects the plane case appear with their solid correlates in Books VI, VII and XI of the *Elements*, so that we may readily assume that the ancients knew the extension to the solid case also. As for higher powers, the impossibility of a geometric representation raises a difficulty. But for any mathematician prepared to manipulate such terms in the flexible manner of Diophantus, both the statement and proof of the general condition of commensurability for any powers would be at hand.[10]

A defect in the Euclidean formulation is worth noting at this point: as it stands, it does not yield a complete criterion for commensurability in length or in (second) power only, for it does not state how a ratio of integers is known to equal or not to equal a ratio of square integers. A lemma attached to this proposition attempts to supply this gap by noting that "similar plane numbers," and only they, will have the ratio of a square number to a square number; but this lemma is recognizable as a post-Euclidean addition.[11] As it happens, one can obtain the needed result from the Euclidean number theory, to show, for instance, that if a ratio equals a ratio of square integers, its least terms are square integers.[12] Perhaps Euclid viewed as obvious this or some equivalent. But we would be more comfortable in our assessment of the formal precision of his theory of irrationals, had he included the appropriate statement and proof of this condition.

We have the following important testimony to Theaetetus' theory of irrationals from the commentary on Book X by Pappus of Alexandria:

> Theaetetus distinguished the powers commensurable in length from those incommensurable, and he distributed the very well known among the surd lines according to the means, so that he assigned the medial line to geometry and the binomial to arithmetic and the apotome to harmonics, as Eudemus the Peripatetic reported.[13]

Interpreting this passage should take into consideration that the informant, Eudemus, was a disciple of Aristotle, hence positioned between the times of Theaetetus and Euclid. The terms "medial", "binomial" and "apotome" are basic within the Euclidean classification of the irrational lines, but we learn from another Aristotelian tract that these names were "only recently" introduced.[14] Thus, we may infer from Eudemus' report not that Theaetetus had established a correlation between the Euclidean classes and the means, but rather that Theaetetus formed his own classes of irrationals as the means of given lines. The correlation with the Euclidean classes is thus Eudemus' manner of characterizing what Theaetetus did in terms more familiar to students of the later theory. Interestingly, the initial distinction between linear and square-only commensurables is here spoken of as one of *powers*, rather than of *lines*. In this, Eudemus adheres more closely to the designation we read in Plato's account, rather than that of Euclid.

From this it is possible to infer what Theaetetus' procedure was: starting from two lines taken as commensurable in square, but not in length, he formed from them in turn their geometric, arithmetic and harmonic means and showed that each of these resulted in an irrational line. We would suppose that the treatment of the first two of these was not significantly different from Euclid's handling of the corresponding medial and binomial lines.[15] For instance, let the given lines be $a$ and $b$, such that $a:b$ is not a ratio of integers, but $a^2:b^2$ is. If we denote as $g$ their geometric mean (so that $g^2 = a \cdot b$), then we claim that $g$ is irrational; i.e., $g^2$ is incommensurable with the square of any rational line.[16] It suffices to show that $g^2:a^2$ (or $g^2:b^2$) cannot equal a ratio of integers. For if it did, then since $g^2:a^2 = a \cdot b:a^2 = b:a$, the lines $b$ and $a$ would be commensurable in length, contrary to their having been taken as incommensurable. If next we set $e = \frac{1}{2}(a+b)$, their arithmetic mean, it follows that $e$ also is an irrational line. For if not, it would follow that $(a+b)^2:a^2$ is a ratio of integers. Since $(a+b)^2:a^2 = a^2 + b^2 + 2a \cdot b:a^2$ and, by assumption, $b^2$ is commensurable with $a^2$, it follows that $2a \cdot b$ is commensurable with $a^2$, or $b$ is commensurable with $a$. This contradicts our initial assumption that $a$ and $b$ are incommensurable.

In the treatment of the third case, the harmonic mean $h$ of the given incommensurable lines $a$, $b$, one would use the relation $a - h:h - b = a:b$ by which the mean is defined; this is equivalent to the relation $h = 2a \cdot b/(a+b)$, that is, $h:b = a:e$, for $e$ the arithmetic mean.[17] Then, if $h$ were rational, $h^2:b^2$ would equal a ratio of integers, so that $e$ would be rational; this last contradicts the irrationality of $e$, as just proved. Now, one may observe another relation derived from that used for $h$ above, namely, $h:a - b = 2a \cdot b:a^2 - b^2$. From this, one can refer results relating to $h$ to others dealing with $a - b$. In the Euclidean theory, the *apotome* irrational is defined as $a - b$, and its irrationality is proved via consideration of the ratio $(a-b)^2:a^2$, parallel to the manner given above for the arithmetic mean.[18] It thus happens that Euclid treats the apotome independently of the binomial and relegates to a postscript the property that any binomial $(a+b)$ and its associated apotome $(a-b)$ have a rational product (namely, $a^2 - b^2$).[19] By contrast, the analogue of this property would be the chief instrument for reducing the harmonic to the arithmetic case within the means-based theory of Theaetetus.

In the absence of further documentation, it is pointless to speculate on the details of Theaetetus' theory. Presumably, he worked out certain analogues to results presented by Euclid, for instance, the uniqueness of the representation of an irrational line as a mean between incommensurable terms. But one thing is entirely clear: Theaetetus could not have attained any results on the irrationality of such lines without use of the complete condition on square-commensurability as held in the Euclidean theory. The commentator Pappus rightly contrasts the restricted Platonic statement of this condition with the general Euclidean statement, observing for instance that the terms $\sqrt{8}$ and $\sqrt{18}$, each incommensurable with a supposed unit term, can be recognized as commensurable with each other only through reference to the Euclidean formulation.[20] But Pappus seems to assign this limitation of the Platonic definition to Theaetetus as well. One perceives that he must here be drawing inferences from the Platonic dialogue, without any independent basis in mathematical or historical sources. For his claim is incompatible with a theory of irrationals of even the most rudimentary sort.

| Euclid's Tenth Book at a Glance<br>Table I: Some Basic Definitions and Results |
|---|
| Two magnitudes are *commensurable* if and only if they have a common measuring magnitude (Def. 1; cf. prop. 2–4); thus, commensurable magnitudes have to each other the ratio of an integer to an integer (prop. 5–8). We will write "$a \mathrm{C} b$" when the magnitudes $a$, $b$ are commensurable with each other, and "$a \not\mathrm{C} b$" when they are not.<br><br>Two lines $a$, $b$ are commensurable *in length* whenever $a \mathrm{C} b$; they are commensurable *in square* whenever $a^2 \mathrm{C} b^2$ (Def. 2). If $a \mathrm{C} b$, then $a^2 : b^2$ is a ratio of square integers (prop. 9). Note that lines may be commensurable *in square only*, that is, whenever $a^2 \mathrm{C} b^2$, but $a \not\mathrm{C} b$.<br><br>We posit a certain line $r$ as "the rational;" then any other line $r'$ is *rational* whenever $r^2 \mathrm{C} r'^2$, but *irrational* otherwise (Def. 3). Note that $r'$ may be rational, yet $r \not\mathrm{C} r'$; this is a critical respect in which the Euclidean notion of "rational" differs from that familiar in modern number theory.<br><br>We call an area $A$ *rational* whenever $A \mathrm{C} r^2$, but *irrational* otherwise (Def. 4). Thus, Euclid's notion of rational *areas* agrees with the modern. |
| Whenever the product of two rational lines $r$, $r'$ forms a rational area, $r \mathrm{C} r'$ (prop. 20).<br><br>Let $a$, $b$ be rational lines, commensurable with each other in square only; then the irrational area formed as $ab$ is called *medial*, and its "side" (i.e., the line $c$ such that $c^2 = ab$) is a *medial line* and is irrational (prop. 21; cf. 23, porism).<br><br>The difference of medial areas cannot be a rational area (prop. 26).<br><br>Let $a$, $b$ be rational lines, commensurable with each other in square only; then $a + b$ is called a *binomial* line and is irrational (prop. 36), and $a - b$ is called an *apotome* and is irrational (prop. 73). |

**II. Advances by Eudoxus.** Distinguished among the geometers in the generation after Theaetetus was Eudoxus of Cnidus. He is known best for his invention of a method of limits for the rigorous measurement of curvilinear plane and solid figures, the basis of Book XII of the *Elements*, and for his working out a general technique of proportions equally valid for commensurable and incommensurable magnitudes, the basis of *Elements* V.[21] The latter already establishes a link with the work of Theaetetus, but certain testimonies indicate a further connection. For the commentator Proclus relates that Eudoxus contributed to the study of the means and that he "advanced through the use of analyses the number of things known about the section, which took their start from Plato."[22] The association here with Plato is obscure, and may be nothing more than a bit of Proclus' neo-Platonizing; but the suggestion that he refers here to some study of the "golden section" seems more fruitful, for it permits a connection with the substantial materials on this topic in the Euclidean *Elements*. The construction of the "section" itself must already have been familiar within the older geometry, for it is crucial for solving the problem of inscribing a regular pentagon in a given circle.[23] But other aspects of this study might well have been discovered at the later time of Eudoxus, for instance, the proof that the segments of a line so divided are irrational.[24] One can see how these very results form a bridge between Theaetetus' theory of irrationals, as just outlined above, and the more elaborate structure in Book X. Let us consider the construction of the "section" and two results on the irrationality of its segments, in order to reveal their significance for the development of the theory of irrationals.

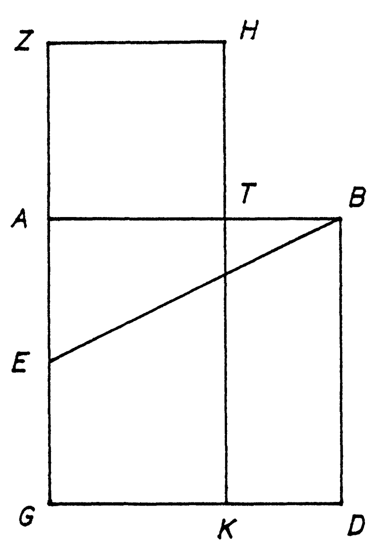


FIGURE 1

The project of the division is to determine the segments $x$ and $y$ of a given line $a$ such that $y^2 = x \cdot a$, or $x : y = y : x + y$. In the Euclidean terminology, the line is said to be divided into "extreme and mean ratio". Following the simpler method in II, 11, rather than the somewhat more complex method in VI, 30, we draw on the given line AB its square ABDG, bisect its side AG at E and join BE. Next, we extend GA to Z such that ZE = EB and complete the square AZHT. Then T divides AB such that the segments AT, TB have the required property; that is, if we set AT $= y$ and TB $= x$, then $y^2 = x \cdot a$.

Without going into the steps of the proof, we may observe the critical feature of the construction, that BE is the hypotenuse of a right triangle whose legs equal $a$, $\frac{1}{2}a$, respectively; that is, $(y + \frac{1}{2}a)^2 : (\frac{1}{2}a)^2 = 5 : 1$. It thus follows that $y + \frac{1}{2}a : \frac{1}{2}a = \sqrt{5} : 1$, whence $y : \frac{1}{2}a\ (= x : \frac{1}{2}y) = \sqrt{5} - 1 : 1$. One thus sees not only that $x$ and $y$ are incommensurable both with respect to $a$ and to each other, but also that $y$ is an apotome irrational (of terms $\sqrt{5}$ and 1). From the relation $x : \frac{1}{2}a = 3 - \sqrt{5} : 1$, it is also clear that $x$ too is an apotome irrational. This is the procedure Euclid follows in establishing the irrationality of these segments in XIII, 6. But one would suppose that a geometer working within the theory of Theaetetus sought to display each of these segments as a mean between two incommensurable lines. We can find these via a method of "false position" based on the expression for the harmonic mean given above. For if $h$, $h'$ are the respective harmonic means of terms $\alpha$, $\beta$ and $\alpha'$, $\beta'$, where $\alpha : \alpha' = \beta : \beta'$, then $h : h'$ has the same value. Let us set $\alpha' : \frac{1}{2}a = \sqrt{5} : 1$ and $\beta' : \frac{1}{2}a = 1 : 1$, so that $h' : \frac{1}{2}a(\sqrt{5} - 1) = \sqrt{5} : 2$. Since $h : \frac{1}{2}a = \sqrt{5} - 1 : 1$, it follows that $\alpha' : \alpha = \beta' : \beta = \sqrt{5} : 2$, whence $\alpha : \frac{1}{2}a = 2 : 1$ and $\beta : \frac{1}{2}a = 2 : \sqrt{5}$. Similarly, since the lesser segment $x$ is such that $x : \frac{1}{2}a = 3 - \sqrt{5} : 1$, it will be the harmonic mean of terms $\alpha$, $\beta$ where $\alpha : \frac{1}{2}a = 2 : \sqrt{5}$ and $\beta : \frac{1}{2}a = 2 : 3$.

In this case, the expression of the greater and lesser segments via the differences $\sqrt{5} - 1$ and $3 - \sqrt{5}$, respectively, is so natural that their alternative expression as harmonic means is hard to suppose other than through the medium of the relation for $h : \alpha - \beta$. This suggests that problems of this kind eventually led geometers to recognize the greater efficiency of dealing with this class of irrationals in the form of the apotome rather than as a harmonic mean, and so to reformulate the theory of Theaetetus in the Euclidean manner.

If we search among the Euclidean results relating to the extreme and mean section, however, only one emerges as of sufficient interest to merit the special attention of a geometer of Eudoxus' caliber. This is the expression of the side of the regular pentagon inscribed in a circle of given rational radius as an irrational line (XIII, 11), by which Euclid expresses as an irrational the side of the regular cosahedron inscribed in a sphere of given rational radius (XIII, 16). Since the manner of inscribing the pentagon (in IV, 10–11) depends on the fact that its diagonal and side are the greater and lesser segments, respectively, of a line divided in extreme and mean ratio, we know from the above that they will be incommensurable with each other, and also that they will be apotome irrationals in the case that their sum is the given rational line. But in the context of the inscription problem, it is the radius of the circle which is the rational line, so that the side and the diagonal will take on new expressions relative to it. Let us consider how a geometer working within the earlier theory would approach the problem of expressing these lines as irrationals. In what follows, I will simplify by replacing the arithmetic and harmonic means with the sum and difference, respectively, of their terms. Euclid investigates only the side of the pentagon in the construction of XIII, 11. While I follow his method, it will be convenient to develop along with it the expression for the diagonal.

Let the regular pentagon ABGDE be inscribed in the circle of center Z and given rational diameter. Draw diameter AZK meeting side GD at right angles at H, join diagonal EB, meeting AK at right angles at T, and join diagonal AG and radius ZG. Let us denote ZG as $r$, a rational line, and set $\mathrm{GD} = s$, $\mathrm{AG} = \mathrm{BE} = d$, $\mathrm{TZ} = x$ and $\mathrm{ZH} = y$. Since $\mathrm{AB}^2 = \mathrm{TA} \cdot \mathrm{AK}$ and $\mathrm{AG}^2 = \mathrm{HA} \cdot \mathrm{AK}$, we have that $s^2 = 2r(r - x)$ and $d^2 = 2r(r + y)$. Further, since the triangles BTZ, AGH are similar (for they are right and the angles at B and at A are equal), $x : r = \frac{1}{2}s : d$; and since the triangles ZGH, BAT are similar, $y : r = \frac{1}{2}d : s$. Since further $d, s$ are in extreme and mean ratio, $d : s = d + s : d$, whence $(s + \frac{1}{2}d)^2 : (\frac{1}{2}d)^2 = 5 : 1$, as seen earlier. It follows that $s : \frac{1}{2}d = \sqrt{5} - 1 : 1$, or that $\frac{1}{2}s : d = x : r = \sqrt{5} - 1 : 4$. Similarly, $y : r = \frac{1}{2}d : s = 1 : \sqrt{5} - 1 = \sqrt{5} + 1 : 4$. Thus, $s^2 : r^2 = 2(r - x) : r = 5 - \sqrt{5} : 2$, while $d^2 : r^2 = 2(r + y) : r = 5 + \sqrt{5} : 2$.[25]

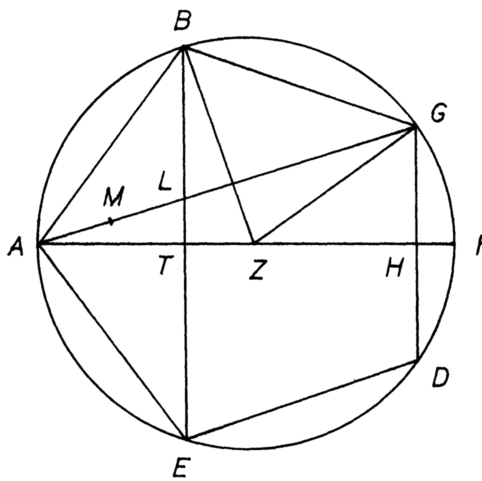


FIGURE 2

This result makes clear that $d$ and $s$ are irrational lines, for $d^2$ and $s^2$ are each incommensurable with $r^2$. But as to what sort of irrational each is, one cannot yet say, since each has been derived not as the sum or difference of incommensurables, but as the square root of such a sum or difference. We have thus to inquire into the nature of the terms $\gamma$, $\delta$ such that $d = \gamma + \delta$, while $s = \gamma - \delta$. The argument will take the form of what the Greeks called "analysis", in that the desired result is at first assumed as known and is then reduced to another result within the set of "givens", that is, of results which have already been established or which can be stipulated.[26] In the present case, we assume that $d$ and $s$ are, respectively, the sum and difference of rational terms $\gamma$ and $\delta$, commensurable with each other in square only—that is, $d$ is a binomial (or arithmetic) irrational, while $s$ is an apotome (or harmonic) irrational—and from this display explicitly what the terms $\gamma$ and $\delta$ are.[27] We must first show that these two assumptions are compatible: that is, that $d = \gamma + \delta$ implies $s = \gamma - \delta$. To do this, we must suppose that the body of the Euclidean theory of the medial lines and areas was already available within the earlier theory. In particular, that an area which is the sum (or difference) of a rational and a medial area is so expressible in only one way (cf. X, 26, 42, 79). Since $d^2 = (r^2/2)(5 + \sqrt{5}) = (\gamma + \delta)^2$, one has that $\gamma^2 + \delta^2 = \frac{5}{2}r^2$ and $2\gamma \cdot \delta = (\sqrt{5}/2)r^2$. Subtracting, we find that $(\gamma - \delta)^2 = (r^2/2)(5 - \sqrt{5}) = s^2$, or $s = \gamma - \delta$, as claimed. Note how the terms $\gamma$, $\delta$ are here set out in two related ways: once via the sum of their squares being given along with their product; and again via their sum and their difference being given separately. From the latter linear relations one readily deduces that $\gamma^2 = (r^2/4)(5 + 2\sqrt{5})$ and $\delta^2 = (r^2/4)(5 - 2\sqrt{5})$.[28]

We have thus displayed $d$ and $s$ explicitly as the sum and difference of incommensurable terms.[29] But it also becomes evident that neither is an irrational line of the standard sort initially supposed. For the lines $\gamma$, $\delta$ would have to be rational, commensurable with each other in square only. Clearly, neither is commensurable in square with $r^2$, so that they are not rational; moreover, if $\gamma^2 : \delta^2 = m : n$, a ratio of integers, it would follow that $m + n : m - n = 10 : 4\sqrt{5}$, and this is impossible since 10, $4\sqrt{5}$ are incommensurable terms. We thus have found that $d$ and $s$ are examples of a new kind of irrational.

Since $d$ is the greater and $s$ the lesser of the segments formed via the extreme and mean division of a line, I will call the former a "greater" irrational and the latter a "lesser" irrational. I believe that the discoverer of this result introduced these very names; for in the Euclidean theory, the classes of irrationals for which $d$ and $s$ may be viewed as the paradigm cases are called, respectively, the "greater" (or "major", *meizōn*) and the "lesser" (or "minor", *elassōn*). The Euclidean names have otherwise entirely eluded the efforts of commentators, ancient and modern alike, to find an explanation.[30]

As for the identity of the discoverer, I have already indicated Eudoxus. As noted, he is said to have advanced the study of "the section" by means of the method of analysis. That is an apt summary of the account just given, while it is difficult to conceive how any other of the Euclidean results on the extreme and mean division could have engaged the efforts of one such as Eudoxus, if indeed that is what is meant by "the section". Moreover, this particular result happens to have crucial significance for the elaboration of the theory of irrationals. This already begins to emerge from what we have seen and will become fully clear from what follows.

**III. Extension of the theory.** Although the diagonal and the side of the pentagon did not turn out to be a simple binomial and apotome, respectively, each was indeed produced as the square root of the product of a rational line by a binomial or an apotome. We are thus led to consider the problem of specifying the conditions under which such a square root will in fact be a simple binomial or apotome. The solution of this problem provides the key to understanding the motive and rationale of the whole Euclidean theory.

Euclid's Tenth Book at a Glance

Table II: Definitions and Relations of Irrationals

Given the rational line $r$, we denote the binomials/apotomes "of order $k$" as $a_k \pm b_k$ and the corresponding additive/subtractive irrationals "of class $k$" as $c_k \pm d_k$. The following relations obtain:

$\sqrt{r(a_k \pm b_k)} = c_k \pm d_k$ (taking signs in same order);

$c_k^2 = \frac{1}{2}r(a_k + s_k)$, $d_k^2 = \frac{1}{2}r(a_k - s_k)$, for $s_k = \sqrt{a_k^2 - b_k^2}$;

$ra_k = c_k^2 + d_k^2$, $rb_k = 2c_kd_k$.

Constructions of the Orders and Classes

| Binomials/Apotomes | | Additive/Subtractive Irrationals |
|---|---|---|
| $k = 1$ | $a_1 \mathrm{C}\, s_1$<br>$a_1 \mathrm{C}\, r$<br>$b_1 \not\subset r$ | $c_1$, $d_1$ rational, $c_1 \not\subset d_1$<br>(hence, $c_1^2 + d_1^2$ is rational,<br>$c_1 d_1$ is medial) |
| $k = 2$ | $a_2 \mathrm{C}\, s_2$<br>$a_2 \not\subset r$<br>$b_2 \mathrm{C}\, r$ | $c_2$, $d_2$ medial, $c_2 \not\subset d_2$<br>$c_2^2 \mathrm{C}\, d_2^2$ (hence, $c_2^2 + d_2^2$ is medial)<br>$c_2 d_2$ rational |
| $k = 3$ | $a_3 \mathrm{C}\, s_3$<br>$a_3 \not\subset r$<br>$b_3 \not\subset r$ | $c_3$, $d_3$ medial, $c_3 \not\subset d_3$<br>$c_3^2 \mathrm{C}\, d_3^2$ (hence, $c_3^2 + d_3^2$ is medial)<br>$c_3 d_3$ medial |
| $k = 4$ | $a_4 \not\subset s_4$<br>$a_4 \mathrm{C}\, r$<br>$b_4 \not\subset r$ | $c_4^2 \not\subset d_4^2$<br>$c_4^2 + d_4^2$ rational<br>$c_4 d_4$ medial |
| $k = 5$ | $a_5 \not\subset s_5$<br>$a_5 \not\subset r$<br>$b_5 \mathrm{C}\, r$ | $c_5^2 \not\subset d_5^2$<br>$c_5^2 + d_5^2$ medial<br>$c_5 d_5$ rational |
| $k = 6$ | $a_6 \not\subset s_6$<br>$a_6 \not\subset r$<br>$b_6 \not\subset r$ | $c_6^2 \not\subset d_6^2$<br>$c_6^2 + d_6^2$ medial<br>$c_6 d_6$ medial |

Let us start with the product $r(a + b)$, for $r$ rational and for $a$, $b$ rational, but commensurable with each other in square only. We intend that the "side", or square root, of this area be a binomial $c + d$; that is, that $c$, $d$ also be rational, commensurable with each other in square only. As before, it follows that $ra = c^2 + d^2$ and $rb = 2cd$. Thus, $r(a - b) = (c - d)^2$, so that we may combine the cases of $a + b$ and $a - b$. Note that one condition has already appeared: that one of the terms of the given binomial, say $a$, must be commensurable with $r$, while the other, $b$, can be commensurable with it in square only. Solving for $c$ and $d$, we find that $c^2 = (r/2)(a + (a^2 - b^2)^{1/2})$, while $d^2 = (r/2)(a - (a^2 - b^2)^{1/2})$. For $c$ and $d$ to be rational lines, as required in the problem, it is necessary that $c^2$, $d^2$ each be commensurable with $r^2$. Since $a$ is commensurable with $r$, we must also have that $(a^2 - b^2)^{1/2}$ is commensurable with $r$. It thus results that the necessary and sufficient condition that the side of $r(a \pm b)$ be, respectively, the binomial or apotome $c \pm d$ is that both $a$ and $(a^2 - b^2)^{1/2}$ be commensurable with $r$.[31]

In the Euclidean terminology, the terms $a \pm b$ are called the "first" binomial and apotome, respectively. If alternatively $(a^2 - b^2)^{1/2}$ is commensurable with $a$, but neither is commensurable with $r$, but $b$ is, the "second" binomial and apotome result; if again $(a^2 - b^2)^{1/2}$ and $a$ are commensurable with each other, but neither they nor $b$ is commensurable with $r$, then the "third" binomial and apotome result. There remain three further cases, where $(a^2 - b^2)^{1/2}$ and $a$ are incommensurable with each other, but $a$ is commensurable with $r$ (the "fourth" binomial and apotome), or $b$ is commensurable with $r$ (the "fifth"), or neither $a$ nor $b$ is commensurable with $r$ (the "sixth"). Clearly, this scheme exhausts in an obvious way the binomials and apotomes in accordance with the solution of the preceding problem. Moreover, for the "first" class, but for none of the derived ones, it holds that the "side" is a binomial or apotome irrational.[32]

It is immediately obvious that the "sides" in each of these derived classes are irrational lines. For if $c \pm d$ were rational, then $(c \pm d)^2$ would be commensurable with $r^2$, whence $a \pm b$ would be commensurable with $r$; but this is excluded, since $a \pm b$ is introduced in all cases as a binomial or apotome irrational. The Euclidean theory bears further marks that this was the manner by which the irrationals beyond the apotome and binomial were first set out. If we consider the particular cases of the diagonal and side of the inscribed pentagon, presented above, since these are found to be the "sides" of the products of the rational $r/2$ by the irrationals $r(5 \pm \sqrt{5})$, respectively, the latter falling under the "fourth" order of binomials and apotomes, all such irrationals formed as the "sides" of lines in this order are called "greater" and "lesser", respectively. The "side" corresponding to the "fifth" binomial is called "that whose square is a rational plus a medial" (for here, $rb$ is a rational area, while $ra$ is a medial area); similarly, the side of the "sixth" binomial is called "that whose square is two medials" (for both $ra$ and $rb$ are medial areas). These have the obvious analogues in the cases of the "fifth" and "sixth" apotomes. Now, the same nomenclature could be applied as well to the "second" and "third" classes. One thus needs an additional identifying feature for these, and Euclid finds it by referring to the terms $c$, $d$ which form their "sides". Since $c^2$, $d^2$ respectively equal $\frac{1}{2}ra \pm \frac{1}{2}r(a^2 - b^2)^{1/2}$, as seen above, and since in the "second" and "third" cases the lines $a$, $(a^2 - b^2)^{1/2}$ are commensurable with each other and simultaneously both incommensurable with $r$, it follows that $c^2$ and $d^2$ are each a medial area, whence $c$ and $d$ are each medial irrational lines. Euclid accordingly designates the second and third classes of additive irrationals the "first" and "second bimedial", with corresponding names for the second and third classes of subtractive irrationals. Note that this designation does truly distinguish between the second and third classes on the one hand and the fifth and sixth on the other. For in the latter classes, the lines $a$, $(a^2 - b^2)^{1/2}$ will be incommensurable with each other, so that their sum or difference will be a binomial or an apotome irrational, respectively, but never a rational line. Thus, for these $c^2$, $d^2$ cannot be medial areas, nor can $c$, $d$ be medial irrational lines.[33]

We close this section with the consideration of a property important within the Euclidean theory: the uniqueness of the representation of any irrational line, or in Euclid's phrase, "an irrational line is divided into its terms in one way only". It is possible to see how this result follows in the general case from its proof for the binomial and apotome. The latter appear in (X, 42 and 79), of which the following is an adaptation. Let us assume the binomial line $a + b$, where $a$, $b$ are rationals, commensurable with each other in square only; and let us suppose another pair of such lines, $a'$ and $b'$, such that $a + b = a' + b'$: it is claimed that $a = a'$, $b = b'$. For if not, since $(a + b)^2 = (a' + b')^2$, the difference $a'b' - ab$ will be a rational area, since the squares of $a$, $a'$, $b$, $b'$ are all rational.[34] Thus, if we set $aa'' = a'b' - ab$, $a''$ will be rational and commensurable with $a$ (cf. X, 20); if we set $ab'' = a'b'$, then $b''$ will be rational, but commensurable only in square with $a$ (cf. X, 22). We thus have $a'' = b'' - b$, so that $b'' - b$ is a rational line; hence, $b''$ is commensurable with $b$ (for otherwise, their difference would be an apotome irrational).[35] Thus, $b$ is commensurable with $a''$, and hence also with $a$; this contradicts our assumption of $a + b$ as a binomial. It follows then that $ab = a'b'$, whence also that $a^2 + b^2 = a'^2 + b'^2$. Thus, $a^2 + b^2 - 2ab = a'^2 + b'^2 - 2a'b'$, so that $a - b = a' - b'$. Since also $a + b = a' + b'$, we have that $a = a'$ and $b = b'$, as claimed. Note that the argument, modified in the obvious way, leads to a proof of the uniqueness of the apotome as well.

Consider now the case of any additive irrational $c + d$, and assume that $c + d = c' + d'$; it is claimed that $c = c'$, $d = d'$. We may express $(c + d)^2 = r(a + b)$ and $(c' + d')^2 = r'(a' + b')$, in accordance with our basic constructions of the irrationals, so that $r(a + b) = r'(a' + b')$. Set $r'a' = ra''$, $r'b' = rb''$; then $a + b = a'' + b''$, an equality of binomials, so that $a = a''$ and $b = b''$, as just shown. Thus, $ra = r'a'$ and $rb = r'b'$, so that $c^2 + d^2 = c'^2 + d'^2$ and $2cd = 2c'd'$. Taking the differences, we have that $(c - d)^2 = (c' - d')^2$, so that $c - d = c' - d'$. Since also $c + d = c' + d'$, it follows that $c = c'$ and $d = d'$, as claimed. As before, the companion theorem for the subtractive irrationals $c - d$ follows in similar fashion.

It is thus clear how the derivation of the irrationals of form $c \pm d$ as the "sides" of areas of the form $r(a + b)$, for $a \pm b$ one of the six types of binomial or apotome, respectively, provides an effective instrument for the proofs of the irrationality and uniqueness of $c \pm d$. We have seen how this same derivation of the irrationals could have arisen in the natural course of early inquiries into this subject. Moreover, their full elaboration within the Euclidean theory, as propositions X, 48–65, 85–102 must surely indicate that Euclid and his predecessors were well aware of these applications of this scheme of derivations. We may now turn to Euclid's formal treatment of these materials in *Elements* X.

**IV. The Euclidean formulation.** One might suppose that any account which provides a natural motivation for the content and nomenclature of a theory like that of Euclid on irrational lines, and which produces straightforward proofs of all of its principal results, would have a strong claim for capturing the manner of the historical genesis of that theory. The account we have just given, however, must confront the central difficulty that Euclid does not in fact present these materials in the manner we have proposed. It is thus necessary briefly to consider Euclid's order of presentation, and then to attempt to explain how and why he diverges from what we should have expected.

The opening section of Book X is devoted to general results on commensurable magnitudes and rational lines. The set of theorems X, 19–26 deals with areas formed as products of rational lines. This includes the result that when the area is rational, its generators are commensurable with each other in length (20), the definition of the "medial" line as the "side" of an area whose generators are commensurable in square only, and the proof that the "medial" is irrational (21), and properties on medial and rational areas, such as that the difference of medial areas cannot be a rational area (26). There follows a section of special constructions (X 27–35) by means of which Euclid goes on to construct the six classes of lines $c + d$ and prove their irrationality (X, 36–41), and later to construct the six classes $c - d$ with their irrationality (X, 73–78). For instance, the "major" irrational (39) is constructed as the sum of lines $c$, $d$ which are taken to be incommensurable with each other in square, but for which $c^2 + d^2$ is rational and $cd$ is medial; how actually to produce such lines $c$, $d$ is known via the earlier problem X, 33. One notes that the problem of determining lines satisfying conditions of this sort is standard within the older field of the geometric "application of areas", embracing portions of *Elements* I, II and VI, to which the constructions in X continually make implicit reference.[36] A prominent feature of the proofs of irrationality here is that each is treated as a special case with its own particular characteristics. While one readily senses the technical similarity of one case to the next, there is decidedly no effort actually to derive any case from any preceding one.

In the next set of theorems X, 42–47 Euclid establishes the uniqueness of the expression for each class of additive irrational, while the analogues for the subtractive irrationals appear in 79–84. Here again the treatments are particular, so that no case is placed in dependence on any other. As it happens, the manner of proof for the third and sixth within each set (i.e., 44, 47 and 81, 84) is comparable to the general method we offered above. But even with this, Euclid makes no attempt to exploit, or even point out, their similarities.

Only at this point does Euclid introduce the six-part division of the binomials and apotomes (in the Definitions preceding X, 48 and 85). Having defined them in the order we followed above, he presents their constructions seriatim in the problems X, 48–53 and 85–90. In the next set of theorems X, 54–59 he shows that the line formed as the "side" of an area generated by a rational and each of the binomials turns out to be one of the additive irrationals already specified; e.g., the "side" associated with the "first" binomial is itself binomial (54), that associated with the "second" binomial is a first bimedial (55), and so on. The analogues for the apotomes and subtractive irrationals appear in X, 91–96. The converse theorems appear in X, 60–65 and 97–102; e.g., that the square of the "major" irrational, when applied to a rational line, produces as width a "fourth" binomial line (63). Although these theorems on the orders of binomials and apotomes in relation to the additive and subtractive irrationals comprise almost a third of the Book, Euclid clearly treats these relations as *properties* of the irrationals, rather than as principal elements in their *definition* and construction. It is in this respect that his order of presentation differs most strikingly from the one we worked out earlier.

Euclid next shows that a line commensurable with one of the additive or subtractive irrationals is itself of the same type, having each of its terms commensurable with the corresponding terms of that irrational (X, 66–69, 103–107). Here again the treatment is particular. To be sure, the two bimedials are presented together (67), as are the two bimedial differences (104); but they are in fact separate cases of a construction which starts off in the same manner, the taking of two medial lines. A faint recognition of the possible interdependence of these theorems arises with the last in each set, in that portions of the proofs can be admitted on the basis of equivalent steps in the theorem preceding. But the fact that all these theorems "are easy and require no elucidation", which leads Heath to omit any further commentary on them,[37] does not deter Euclid from providing a complete, and in most cases quite lengthy, demonstration for each. Now, in the theorems which open each set Euclid establishes not only that lines commensurable with a binomial or an apotome are themselves binomial or apotome, respectively, but also that they will fall within the same one of the six orders of binomial or apotome as that to which they are assumed commensurable (X, 66, 103). On the basis of this result, one can establish all of the remaining results as mere corollaries via the representation of the irrationals as "sides". For let $z$ be commensurable with the irrational line $c + d$; we set $(c + d)^2 = r(a + b)$ for a rational line $r$ and a binomial $a + b$ in one of the six orders. Then if we take $r' : r = a' + b' : a + b = z : c + d$, it follows that $z^2 = r'(a' + b')$ where $r'$ is rational and commensurable with $r$ and $a' + b'$ is a binomial of the same order as $a + b$ (from X, 66); thus, $z$ is an irrational of the same class as $c + d$. Clearly, the same argument applies for the subtractive cases $c - d$. Once more, Euclid has failed to exploit the considerable advantage in efficiency of proof possible through the "side" representation of the irrationals.[38]

In the next set of theorems, Euclid assumes the sum first of a rational and a medial area (71) and then of two medial areas (72), showing how in the former instance the "side" is one of four of the kinds of additive irrationals, while in the latter it is one of the remaining two kinds. We have already remarked how this property is implied in the names of the fifth and sixth irrationals (i.e., "that which produces in square a rational and a medial area", and "that which produces in square two medial areas", respectively). In all these cases the property is evident from the representations of the irrationals as "sides" associated with the various binomials, and Euclid does in fact here frame the proofs around those representations. The analogous relations for differences among rational and medial areas are given in X, 108–110; where only two cases arose relative to the additive irrationals, three occur here, owing to the asymmetry in the instance of the difference between a rational and a medial area. Each of the three cases corresponds to two of the classes of irrationals.

The section on the additive irrationals ends in a scholium after X, 72, to the effect that each type of irrational is different from the others; that is, no line can belong to more than one of the six additive classes. This is proposed as evident, since each is a side associated with one of the orders of binomial, while the latter are obviously distinct from each other by virtue of the manner of their definition.[39] The parallel observation is made for the subtractive irrationals in a scholium after X, 111. That theorem itself establishes that no apotome can equal a binomial, so that one obtains the basis for the summary claim that the thirteen classes—the medial, the six additive and the six subtractive—form a disjunctive division of the irrational lines.

This is plainly a good place to end the theory. But Euclid appends four more theorems. Three of these (X, 112–114) deal with the products of cognate binomial and apotome lines; e.g., that if $a + b$ is a binomial and $a' - b'$ an apotome such that $a$ is commensurable with $a'$, $b$ with $b'$ and $a : a' = b : b'$, then their product is rational (X, 114). This ought to pose no difficulty at all. For $(a + b)(a - b) = a^2 - b^2$ (cf. II, 5, 6), which is rational, being the difference of rational areas. Moreover, $a' - b' : a - b = a' : a$, so that $(a + b)(a' - b') : (a + b)(a - b) = a' : a$. Since the latter are commensurable, $(a + b)(a' - b')$ will be commensurable with the rational area $(a + b)(a - b)$, hence will itself be rational, as claimed. Euclid's proofs are monstrously complicated; it is left as an exercise for the reader to figure out why. But his failure to take up the cases of the products of the other cognate irrationals is more easily understood. If we consider the simple case of $(c + d)(c - d) = c^2 - d^2$, the relations we developed above for $c^2$, $d^2$ show that $c^2 - d^2 = r(a^2 - b^2)^{1/2}$. Now, in the first class (the binomial and apotome), the radical is taken commensurable with $a$, the latter being commensurable with $r$, so that the radical is also commensurable with $r$, whence the product is rational. In the second and third classes, the radical is also commensurable with $a$, but $a$ is incommensurable with $r$; hence the product is medial. In the fourth class, the radical is incommensurable with $a$, the latter being commensurable with $r$, so that the product is again medial. As for the last two classes, the radical here is incommensurable with $a$, while $a$ is incommensurable with $r$; but this does not permit in general a determination as to whether the radical is commensurable with $r$ or not. We thus cannot say here whether the product is rational or medial. In effect, then, one lacks a clear-cut theorem to state for the general case of products of cognate irrationals. This would be sufficient reason, I suppose, that Euclid did not try to raise the issue.[40]

In the last theorem (X, 115) Euclid shows that the procedure of forming medial lines and medials of medials and so on gives rise to a never-ending sequence of new irrational classes. This result was later generalized by Apollonius, who showed that the interpolation of any number of mean proportionals between two given rational lines, commensurable in square only, yields irrational lines. We recall that this extension is prefigured in the work of Theaetetus; for note that he considered the incommensurability of cubic roots, while these result from the construction of two mean proportionals between given lines. It is clear that Euclid's closing theorem suddenly throws open the field of inquiry, after the main body of the theory had been neatly tied and sealed in the scholium to X, 111 only four theorems earlier. This may be good cause for viewing the result as a post-Euclidean addition, as the prominent editors Heiberg and Heath do.[41] Heath resists Heiberg's similar suspicions about the product theorems in 112–114, and I believe rightly so.[42] Rather than looking forward to a new inquiry, they impress me as a remnant of the early stage of the theory, where the irrational lines were formed as geometric, arithmetic and harmonic means. Euclid's product theorems are analogues of the proportionality linking the means (i.e., $a : g = g : h$), the latter being an indispensable instrument for Theaetetus' theory, but later relegated to a subordinate status in the course of the revision and extension of the theory. Notably, when the commentator Pappus takes up the question of the general expression of the additive irrationals as arithmetic means and the subtractive as harmonic means, in order to complete Theaetetus' results for the binomial and apotome, he makes continual use both of the proportionality of the means and of the product theorems for the cognate irrationals.[43] The latter he recognizes to hold not only for the binomial and apotome, as in X, 112–114, but for the other classes as well, much as we set them out above. Thus, whatever the actual provenance of this concluding set of theorems in Book X, their relevance for the theory even in its pre-Euclidean phases is unmistakable.

This review of the Euclidean theory exposes a remarkable feature: that the imposing and carefully worked out structure of theorems in Book X belies the inherent simplicity of this subject matter through its presentation of obscurely motivated constructions and unnecessarily cumbersome proofs. In particular, the failure to exploit the "side" representations of the irrationals compels Euclid to produce separate treatments of a dozen special cases for each of the main results on the irrationality, the uniqueness and the closure (relative to linearly commensurable terms) of the irrational classes. These defects in technical execution must have been so obvious to any participant in the development of the theory, that we must assume their presence in Euclid's work were somehow the necessary consequences of decisions made on the organization of the theory. It is a notorious aspect of the classical geometry, and surely of other periods in mathematical history, that the order of formal presentation of results frequently, if perhaps not invariably, alters the order of discovery. Indeed, the formal "syntheses" which secured the solutions found via "analyses", in accordance with the standard technique of ancient geometry, typically reverse the order of the initial treatment. In a similar manner, an editor of the theory of irrationals might well view the deduction of the forms of the irrationals, via their constructions as the "sides" of areas associated with the six types of binomials and apotomes, as if it were an analytic preliminary to the formally preferable manner of their construction: namely, as lines formed directly as the sums or differences of terms $c$, $d$ given their own detailed construction. In effect, defining the lines implicitly via the forms of their squares might be deemed less satisfactory than the latter explicit constructions of the lines themselves, even if this turns out to be a more complicated procedure. Once that decision has been made on the ordering of the formal constructions, however, the difficulties in proof technique follow almost inevitably.

The shift from heuristic analysis to formal synthesis subtly transforms the objective motivating the presentation of the theory. In the former instance, one would include the discovery of the forms of the irrationals and the efficient management of the techniques of their application as important goals. But in the latter case, one aims at a comprehensive exposition of a subject matter known, each result to be secured via a fully detailed proof. The ideal of comprehensiveness lays a new burden on certain aspects of the presentation. Ever since the Danish mathematician and historian H. G. Zeuthen proposed the thesis almost a century ago,[44] one has commonly viewed the problems of geometric construction, such as one finds throughout the *Elements* and other ancient treatises in geometry, as a genre of existence proofs to justify the introduction of the special entities required in subsequent proofs. Accordingly, it is sometimes maintained that the problems of construction which Euclid solves in X, 27–35, 48–53 and 85–90 (e.g., 27: "to find medial lines commensurable in square only which contain a rational area"; 48: "to find the first binomial"; 85: "to find the first apotome") are intended as existence proofs supporting the constructions and proofs of the additive and subtractive irrational lines (36–41; 73–78) and the theorems on their relations to the orders of binomials and apotomes (54–65, 91–102). But surely this view is incorrect. If Euclid wished merely to show that lines exist satisfying the specified conditions, he need only have provided a particular example of the construction at issue. For instance, the diagonal and side of the inscribed regular pentagon would serve easily and admirably to demonstrate the existence of "major" and "minor" lines and the associated "fourth" binomials and apotomes. Instead, Euclid effects the constructions in general, thus obtaining an explicit procedure for producing *every* term in *every* class of irrationals.[45] I believe that nothing short of this could answer the purposes of his formal system of the theory.

In view of this, there is a surprising lapse relating to one of the constructions, which, as far as I know, has not before now been pointed out by scholars on this subject. The problems in X, 29–35 on which the constructions of the irrational lines depend refer in their turn to two constructions given in lemmas just preceding X, 29. Their object is to find two square numbers (integers) whose sum, in the first case, is also a square number, but in the second case is not. Now, if Euclid wished only to establish the *existence* of "Pythagorean integer triples," he need only have noted that $3^2 + 4^2 = 5^2$ and been done with it. Instead, he establishes the form which *all* such integer triples must satisfy. His construction starts with an arbitrary pair of "similar plane numbers" $m$, $n$ (i.e., admitting of a factorization $m = ab$, $n = cd$ such that $a : b = c : d$; whence, $m : n = a^2 : c^2$, a ratio of square integers; cf. VIII, 26 and its use in the lemma after X, 9), where $m$, $n$ are either both even or both odd, and then forms $p = \frac{1}{2}(m + n)$, $q = \frac{1}{2}(m - n)$. He then shows that $p^2 = q^2 + mn$; and since $mn$ equals a square number (i.e., $a^2d^2$), this secures what is asked in the first lemma. In a corollary, he observes that if $m$, $n$ are not of the specified kind, the difference $p^2 - q^2$ will not be a square integer (for if $m$, $n$ are not similar plane numbers, their product will not be square; cf. IX, 2). Euclid has thus provided the necessary and sufficient conditions that the sum of square numbers be a square number.

It is in the second lemma that difficulty arises. Here one seeks square numbers whose sum is not square. As before, Euclid takes $m$, $n$ as similar plane numbers, both odd or both even, forms $p$ and $q$, but considers the sum $(q - 1)^2 + mn$; in a complicated proof, he shows that each of the three possibilities—that the sum be greater than, equal to or less than $(p - 1)^2$—reduces to a contradiction; hence, the desired condition of the lemma is satisfied. Now, by contrast with the preceding lemma, this one does not effect the general construction; thus, for the strict logical requirements of the theory it is hardly better than merely observing that $2^2 + 4^2 = 20$, whence at least one knows that numbers of the specified sort exist. Not only is the construction inadequate for his purposes, but even more puzzling, for all its complexity it is unnecessary. For the general condition can easily be inferred from the preceding lemma. We need only choose an arbitrary square number $r^2$ and consider *all* of its factorizations $r^2 = mn$, such that $m$, $n$ are both odd or both even (note that IX, 2 entails that $m$, $n$ are similar plane numbers); it is obvious that for any $r^2$ there are only finitely many different factorizations. Corresponding to each of the pairs $m$, $n$, we form the value $q = \frac{1}{2}(m - n)$, and then take $q'$ any integer different from all these values of $q$. It follows that $r^2 + q'^2$ cannot be a square integer. For if, say, $r^2 + q'^2 = p'^2$, the construction in Euclid's previous lemma shows that $q' = \frac{1}{2}(m - n)$ for certain integers $m$, $n$ such that $mn = r^2$. Hence, $q'$ must assume one of the values explicitly excluded in the construction. This contradiction thus establishes that $r^2$, $q'^2$ fulfil the condition of the lemma. In this alternative form, then, the construction yields the complete condition needed for the theory, and does so without any technical elements beyond those already worked out in the first lemma. Euclid's handling of the second lemma is thus yet another unsettling reminder that the proofs in Book X are not always as well conceived as they commonly are purported to be.

**V. Assessment.** Although I have continually referred to the theory in Book X as being Euclid's, the question of provenance remains to be considered. It is of course possible that the Book is Euclid's own composition, consolidating the results of two generations of research on irrationals. But it seems equally possible that it is merely his own edition of a complete treatise on irrationals written by an earlier geometer. We have seen that the conceptions which initiated the study of irrational lines were due to Theaetetus in the first third of the fourth century B.C. But if my view of his approach is correct, and if indeed the important insights on the irrationality of the diagonal and side of the pentagon are due to Eudoxus, then we can hardly assign to Theaetetus the completed theory of Book X. Moreover, in view of the formal defects which mar that treatment of the theory, we would surely hesitate to assign it to Eudoxus or a comparably gifted disciple. We learn through the commentator Proclus the names of a few geometers active in the period between Eudoxus and Euclid.[46] Among these, one Hermotimus of Colophon is said to have "advanced further the things already well provided by Eudoxus and Theaetetus and discovered many of the elements [sc. things presented in the *Elements* of Euclid] and composed some things concerning loci." If this statement can be accepted at face value, it conforms well with the assignment of the formal composition of the theory of irrationals in Book X to Hermotimus; for one supposes with difficulty that any other portion of the *Elements* remained to be *discovered* in the period after Eudoxus. But in view of the meagerness of our information on this period, we must admit the possibility that not even the name of the responsible individual has been transmitted in our sources.

The relevant issue, then, is whether to assign authorship of Book X to Euclid himself or to place it with some such geometer as Hermotimus in the preceding generation. Our decision can be made only through a consideration of the motives and character of the author, as implicit in the Book itself, and of whether that portrait is compatible with our image of Euclid. The most striking feature to account for, as we have seen, is the author's failure to exploit the "side" representations of the irrationals, so that he unnecessarily burdens the proofs. We might dismiss the other lapses we have noted, such as the unwieldy demonstrations of the product theorems in X, 112–114 and the inadequate solution to the construction problem in the lemma prefacing X, 29, on the grounds that these might be clumsy additions by post-Euclidean interpolators. But this other matter of the handling of the irrationals as "sides" lies at the heart of the conception and organization of the theory. Is it possible that Euclid could be culpable of such defects in proof technique? Alas it is. For with respect to another (now lost) Euclidean work, the *Porisms*, Pappus finds it possible to consolidate ten of Euclid's enunciations into a single proposition and in this way to recognize an even more general configuration of which they are special cases.[47] The very fact that Euclid chose to transmit Book X in its extant form seems to testify to his basic sympathy with the approach therein adopted.

Nevertheless, I perceive a mitigating factor which justifies shifting responsibility away from Euclid. Simply put, Book X is a pedagogical disaster. There might be a certain interest in seeing how a general construction admits of modified treatment in each of its particular cases; sometimes (although rarely in Book X) the result is a more elegant way of treating the case than the general procedure would be. But surely this is a concern for the formalist expositor of a theory, not the teacher of mathematical techniques. In view of the likely genesis of the theory, we can be certain that the author well knew, for instance, that the side of the regular pentagon exemplifies the "minor" irrational. But he does not in fact inform us of any such "real" configurations which might remedy the apparent artificiality of his inquiry. The essential mathematical ideas are smothered in detail, while the interest to develop skill in the implementation of these techniques in geometric contexts is set at far remove.[48] The history of the study of Book X provides evidence of these weaknesses. Among the ancients, Pappus knows of two other configurations which give rise to an irrational line,[49] and he sketches a few extensions made by Apollonius, the one generalizing the medial line via the interpolation of any number of mean proportionals, the other considering trinomial and quadrinomial expressions, etc., on the model of the binomial and the apotome.[50] But Pappus' other observations on the construction of the Euclidean irrationals as means are of no great profundity, while other commentators cannot report any further results developing out of this theory.[51] In this sense, one modern judgment of the work as "a mathematical blind alley" is quite apt.[52] Certain potentials of the subject matter are entirely missed, for instance, insight into how products of the form $(a \pm b\sqrt{N})(c \pm d\sqrt{N})$ bear on the finding of solutions to integral relations of the form $x^2 - Ny^2 = \pm m$.[53] Indeed, the task of merely comprehending the Euclidean theory itself seems to demand that one forsake its manner of exposition, and instead, in Stevin's phrase, to search "through its causes." If the subject is intrinsically "easy and without difficulty," its treatment in Book X will be found "incomprehensible" or "obscure," or misjudged as "profound," by the commentator who attempts a direct frontal assault on the formal theory.

The student who approaches Euclid's Book X in the hope that its length and obscurity conceal mathematical treasures is likely to be disappointed. As we have seen, the mathematical ideas are few and capable of far more perspicuous exposition than is given them here. The true merit of Book X, and I believe it is no small one, lies in its being a unique specimen of a fully elaborated deductive system of the sort that the ancient philosophies of mathematics consistently prized.[54] It constitutes the results of a detailed academic exercise to codify the forms of the solutions of a specific geometric problem and to demonstrate a basic set of properties of the lines determined in these solutions. One can thus profitably study Book X to learn how its author sought to convert a body of geometric findings into a system of mathematical knowledge.

But our Euclid is not the author of obscure research monographs; he is the master pedagogue, the author most notably of the *Elements*, the most effective technical textbook ever written. If we hold to this stereotype (what choice have we?), then it is difficult to view him as having actually composed Book X. It is difficult enough even to fathom how he could have allowed this subject to be transmitted in such an unwieldy form, save perhaps as a formal challenge to students who had completed their introductory courses in plane geometry (Books I–VI) and number theory (Books VII–IX). But one would surely prefer to suppose that any effort on his own part to elucidate the theory of irrationals would have resulted in a clearer exposition than Book X. Or might Euclid have perhaps thought that the experience of confronting such a dense treatment would stimulate his students to discover alternatives through a recourse to the fundamental ideas—in effect, to respond as Stevin and other commentators would do much later?[55] We may at least entertain in passing this somewhat romantic notion, despite the hint of perversity it seems to project onto Euclid. But if it happened to be true, it was a secret well kept by (or from) the ancient commentators, who for all their efforts failed to discern any more convenient alternative.

It seems far more likely that Euclid merely incorporated into the *Elements* a complete treatise on the irrationals prepared by a geometer not much earlier than he. This would account for the omission from Book X of one of the results needed for Book XIII (see footnote 38); for the entire conception of Book X requires only an awareness of the values for the diagonal and side of the inscribed regular pentagon, as paradigms respectively of the "major" and "minor" irrationals, in relation to the basic classes of "binomial" and "apotome." Thus, the specific application of the scheme of irrationals toward the constructions of the regular solids, as in Book XIII, could come later and thus implement findings not included in this particular synthesis of the theory. It is of course a lapse on Euclid's part not to have caught and corrected this inconsistency between the two Books; but surely it is more difficult to imagine that he could have composed them in this defective manner. This view indicates further that Book X need not be taken as the full measure of the knowledge of irrationals at Euclid's time. Note that when Pappus works through his theorems generalizing on the formation of the subtractive irrationals as harmonic means, he sometimes speaks as if this were in fact a theorem of Theaetetus.[56] The theorems themselves are not likely to have attracted research interest very long after the special definitions of Theaetetus' theory had receded into the historical distance. But they are quite appropriate to the concerns which would arise when the "Euclidean" classes of irrationals were introduced and their properties worked out. Thus, Pappus' ambiguity as to what Theaetetus did could well be transmitting a lack of clarity on the part of his source, Eudemus, to separate Theaetetus' results from their subsequent extensions. The actual treatise taken over by Euclid omitted some of these extensions, doubtless for their not yet being entirely familiar at the time of its composition. One expects an interval between the discovery of results and their incorporation into a textbook literature. But the most striking feature of Book X is surely how its author encumbers the proofs through his interests in system. By laying down this "mathematicians' cross" he demonstrates with extreme effectiveness how greatly the rigors of synthetic exposition can be in conflict with the heuristic aims of analysis.

## NOTES

[1]My translation from the *Arithmétique*, 1585, p. 161; cf. p. 164 and the "Appendice" surveying the contents of Book X, pp. 187 ff. Heath [1926, III, 8f] cites from a later edition (*Oeuvres*, 1634, pp. 218–222) a remark to similar effect (my translation from the French): "The difficulty of the Tenth Book of Euclid has become for many in horror, even to calling it the cross of the mathematicians, a subject matter too hard to digest and in which they perceive no utility." Stevin (1548–1620) was noted as an engineer and writer on mathematics. His *Principal works* have been edited by D. J. Struik (4 vols., Amsterdam, 1955–64); and his influential tract on decimal computation, *De Thiende*, has been issued in a German edition by H. Gericke and K. Vogel (Frankfurt am Main, 1965).

[2]Heath [1926, III, 4f] cites the "valuable remark" by H. G. Zeuthen, that since the Greeks solved equations geometrically, and "inasmuch as one straight

line looks like another...it was necessary to undertake a classification of the irrational magnitudes which had been arrived at by successive solution of equations of the second degree." He mentions several other treatments of a comparable nature (*ibid.*, p. 8) and emphasizes this approach in his own account of the Book (see note 32 below). G. Junge [1930, 17–26] seems much closer to a viable interpretation when, following the view of Chasles, he attempts to sever this "Gordischer Knote" (p. 19) via the project of simplifying surd expressions. Such a view, as refined by van der Waerden (see note 4), is not greatly different from the one I will go on to present.

[3]I. Mueller [1981, Chapter 7.2] provides an immensely detailed account of Euclid's exposition. But even he admits to finding aspects of Euclid's theory "not completely perspicuous" or without "clear intuitive motivation" (p. 268). He rightly rejects assigning an "algebraic motivation" for the reasoning of Book X; but he reacts too strongly, I believe, in denying that it has "a clear mathematical goal intelligible to us in terms of our notions of mathematics" and in maintaining that "book X has never been explicated successfully in [such a] way nor does it appear amenable to explication of this sort" (pp. 270 f). He himself views it as the formal elaboration of a classification of irrational lines conceived in response to the construction of the icosahedron in XIII, 16. As my discussion in §II will show, I too wish to assign this construction a role in the genesis of the theory. But the theory itself, as I show in §III, evolves around another problem, one closer to that singled out by Junge and van der Waerden (see next note).

[4]Knorr [1975, 279–284]; van der Waerden [1963, 168–172]. Comparably to Chasles and Junge (see note 2 above), van der Waerden frames the structure of Book X around the problem of determining when the root of a certain irrational is an irrational of the same kind (cf. the discussion of the binomials and apotomes in §III below). The reader will find his breakdown of its structure extremely lucid, but may be puzzled at his insistence that "the line of thought is...purely algebraic" (p. 171).

[5]I discuss the early studies in detail in Knorr [1975, Chapter 2].

[6]For the construction, see Knorr [1975, 26–28]. The Aristotelian passages are set out by Maracchia [1980], commented on by Knorr [1981].

[7]For instance, a Babylonian tablet from the mid-second millennium B.C., in writing for $\sqrt{2}$ the sexagesimal value 1; 24, 51, 10, testifies to the early computational experience with such quantities; cf. Neugebauer [1957, 35, 50].

[8]Theaetetus 148 a-b; for a full discussion of the passage, see Knorr [1975, Chapter 3].

[9]*Elements* X, 9; a scholium assigns this theorem to Theaetetus (cf. Knorr [1975, 64, 97]).

[10]For an account of Diophantus' notation and a survey of the *Arithmetica*, see Heath [1921, II, 456 ff]. Diophantus introduces a terminology of powers extending indefinitely, and actually includes manipulations of higher powers, like the fourth and the sixth, in some of his problems (cf. *ibid.*, pp. 506 f).

[11]Heath [1926, III, 31; cf. II, 383].

[12]For a proof, see Knorr [1975, 232 f].

[13]Pappus, *Commentary*, §I, 1; I translate from the Arabic text, ed. Thomson (Pappus [1930, 192]). The description of these three irrationals as "very well-known" (*al-mašhūra jiddan*) seems puzzling, and Thomson (who renders it "more generally known", p. 63) does not try to explain it. I would suppose the Arabic translator saw a term like *gnōrimōterai*. If so, the sense would be not that these lines are "familiar," but that they are "special" in relation to the others. In effect, Eudemus would be signifying that these three cases were set off from the others. Note that the property at issue, the expression of the irrational lines as means, is known to Pappus in a form that embraces the other ten classes as well (*Commentary* II, 17 ff; cf. note 40 below). We would thus infer that the generalization had not yet been worked out by Theaetetus.

[14]*On Indivisible Lines*, 968 b 19: "the other [irrational lines] which *have recently been discussed*, such as the apotome or the binomial." I follow the defense by Heath [1949, 255 f] of this rendering of the phrase in emphasis, against those who have proposed textual emendations. Heath seems not to perceive, however, that this reading is incompatible with his view that the names are due to Theaetetus. For there is a gap of over a half-century separating Theaetetus and the tract, so that its author could hardly refer to such studies as "recent." The anomaly disappears when one accepts that the terms "binomial" and "apotome" were introduced through the later development of Theaetetus' theory.

[15]*Elements* X, 21 and 36, respectively. I adopt an indirect method of proof, where Euclid follows a direct method.

[16]On the definition of "rational," see Table I.

[17]Pappus cites a form of this relation in his *Commentary*, II, 18 [1930, 141]; he gives its geometric representation in *Collection* III [1876, I, 70]. Among the arithmetic writers, it was called the "perfect proportion;" cf. van der Waerden [1963, 94] and Heath [1921, I, 85 ff].

[18]*Elements* X, 73.

[19]Cf. X, 114.

[20]*Commentary*, I, 10 [1930, 72–74].

[21]On Eudoxus, see Knorr [1975, Chapter 8, §4] and van der Waerden [1963, 179–190]. I have proposed a revised view of Eudoxus' contribution to the theory of proportions in Knorr [1978].

[22]Proclus [1873, 67].

[23]*Elements* IV, 10–11 via II, 11.

[24]XIII, 6.

[25]For more detailed accounts of the construction, see Heath [1926, III, 461–466] and Mueller [1981, 260–263].

[26]The definitive ancient statement of the method of analysis is by Pappus, *Collection* VII, preface [1876, II, 634–636]. This method is of central interest within my forthcoming study, *The ancient tradition of geometric problems* (Birkhäuser).

[27]If in Figure 2 lines EB, AG meet in L and AL is bisected at M, then since GL $= s$, we have that GM $= \gamma$ and MA $= \delta$.

[28]Since $d^2 : s^2 : r^2 = 5 + \sqrt{5} : 5 - \sqrt{5} : 2$, we have that $d^2 + s^2 : 2sd : r^2 = 10 : 4\sqrt{5} : 2$, whence that $(d \pm s)^2 : r^2 = 5 \pm 2\sqrt{5} : 1$. Since $\gamma$, $\delta$ respectively equal $\frac{1}{2}(d \pm s)$, the stated expressions for $\gamma^2$, $\delta^2$ follow.

[29]Note also that just as $d$ is twice the arithmetic mean of its terms so also $s$ is twice their harmonic mean. For $h : \gamma - \delta = 2\gamma\delta : \gamma^2 - \delta^2 = \frac{1}{2}\sqrt{5} : \sqrt{5}$. As one can infer from the relations in Table II, this property of the harmonic mean entails that $a^2 = 5b^2$, so that any line $c - d$ for which it holds must be commensurable in square with the line $s = \gamma - \delta$ (and also $\gamma : \delta = c : d$; cf. note 38). One suspects that this special feature of this construction of $s$ was a factor in the transition from the harmonic irrationals of Theaetetus to the apotomes of Euclid.

[30]Heath [1926, III, 7, 88, 164] and van der Waerden [1963, 172] do not even raise the issue; Mueller [1981, 274] writes: "I know of no satisfactory explanation for the term 'minor'." An ancient scholiast proposes that Euclid named the "major" by virtue of the fact that the rational area $c^2 + d^2$ is greater than the medial area $2cd$ (see Table II) "and it is necessary to order the naming from the speciality of the rationals" (Euclid [1969-, III, ?25]). While the claim is true (he gives a proof), it hardly accounts for the naming of the "minor," where the same property holds.

[31]While it is obvious that $c$, $d$ are the positive roots of the biquadratic equation $x^4 - rax^2 + \frac{1}{4}r^2b^2 = 0$, facts of this sort would seem of little use in piecing out the objectives underlying Euclid's theory. Nevertheless, many interpretive efforts have tried to utilize just such relations (see next note).

[32]One thus notes the prominence of the relation given by Chasles as the formula: $\sqrt{A \pm B} = \sqrt{\frac{1}{2}A + \frac{1}{2}(A^2 - B^2)^{1/2}} \pm \sqrt{\frac{1}{2}A - \frac{1}{2}(A^2 - B^2)^{1/2}}$, and can agree with Junge [1930, 23] who styles it the "quintessence of Book 10". Heath follows other commentators in attempting a more ambitious algebraization (cf. note 2). He first represents the six binomials and apotomes as the roots $x$ of the quadratic $x^2 \pm 2\alpha x \cdot \rho \pm \beta \cdot \rho^2 = 0$, so that the irrational lines $x'$ (for which $x'^2 = x \cdot \rho$) become the roots $x$ of the biquadratic equation $x^4 \pm 2\alpha x^2 \cdot \rho^2 \pm \beta \cdot \rho^4 = 0$ (Heath [1926, III, 5–7 and *passim*]). For instance, the binomial/apotome $\rho \pm \rho\sqrt{k}$ (for $k$ a nonsquare rational number) corresponds to $\alpha = 1 + k$ and $\beta = (1 - k)^2$. But this scheme hardly lays bare Euclid's motives. Why, for instance, would he wish to make these choices for the coefficients? Indeed, why would he be interested in classifying the roots of *biquadratic* equations in the first place?

[33]These relations are summarized in Table II. In view of Theaetetus' adoption of the means for defining the irrational lines, the following problem is likely to have been considered within the development of the theory: under what conditions is the irrational line $c - d$ commensurable with the harmonic mean of $c$ and $d$? Since $h : c - d = 2cd : c^2 - d^2 = rb : r(a^2 - b^2)^{1/2}$, it follows that $b \mathrm{C} (a^2 - b^2)^{1/2}$, whence that $a \not\mathrm{C} (a^2 - b^2)^{1/2}$ (for $a \not\mathrm{C}\, b$). If we set $b : (a^2 - b^2)^{1/2} = m : n$, then $b^2 : a^2 = m^2 : m^2 + n^2$, where $m^2 + n^2$ cannot equal a square integer (cf. the discussion of the lemmas to X, 29 at the end of §IV below). There will be cases where this holds among the 4th, 5th and 6th classes of subtractive irrationals. Note that $b^2 : a^2 = 1 : 5$ corresponds to the irrational side of the regular pentagon inscribed in the circle of rational radius (cf. note 29 above).

[34]In X, 42 Euclid can assume from X, 26 that the difference of medial areas cannot be a rational area; the rest of my proof here thus effects the result of that theorem.

[35]Euclid proves the irrationality of the apotome in X, 73, so cannot use this result in X, 26, but rather must provide an argument effectively duplicating a portion of the later proof. Since I have already established the irrationality of the apotome, it can be assumed here.

[36]On this technique see Knorr [1976, Chapter 6, §4] and van der Waerden [1963, 118–124]. Some form of these methods was already familiar within Greek geometry in the latter part of the fifth century B.C.; their strong affinity with methods from the much older Mesopotamian metrical geometry discourages the view of their independent invention by the Greeks.

[37]Heath [1926, III, 147].

[38]Mueller notes [1981, 283, 299] that Euclid must assume a stronger result when he later claims that the sides of the icosahedron and dodecahedron inscribed in the same sphere (assumed of rational radius) are incommensurable with each other in square (XIII, 18): namely, that if two irrational lines are commensurable with each other in square, they will fall within the same class of irrationals. (Thus, since the sides are, respectively, a minor and an apotome, they cannot be commensurable in square.) We may observe that yet a stronger claim can be established: that if $c \pm d$, $c' \pm d'$ (taking signs in the same order) are irrational lines commensurable with each other in square, then $c : c' = d : d'$. We need only consider the additive case: setting $(c + d)^2 = r(a + b)$ and $(c' + d')^2 = r(a' + b')$, since $c + d$, $c' + d'$ are commensurable in square, $a + b$ and $a' + b'$ will be commensurable with each other in length; hence, by the theorem on commensurable binomials (X, 66) $a + b$, $a' + b'$ will fall within the same order of binomials, for $a : b$ will equal $a' : b'$. Thus, $c + d$ and $c' + d'$ will fall within the same class of additive irrationals. Moreover, since $(c + d)^2 : (c' + d')^2 = a + b : a' + b' = a : a' = b : b'$, and $c^2 + d^2 : c'^2 + d'^2 = ra : ra'$, and $2cd : 2c'd' = rb : rb' = a : a'$, it follows after subtracting that $(c - d)^2 : (c' - d')^2 = a : a'$. Thus, $c - d : c' - d' = c + d : c' + d'$, whence $c : c' = d : d'$, as claimed.

[39]Heath [1926, III, 158] incorrectly supposes that in the scholium after X, 72 Euclid also means to show the distinctness of the orders of the binomial. But in fact, this result is obvious from their definitions; Euclid states it here in order to establish the distinctness of the associated classes of irrational "sides".

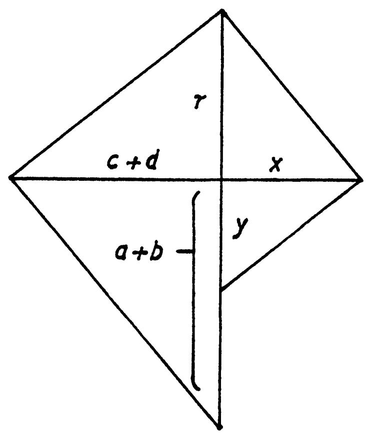


FIGURE 3

[40]Pappus somewhat finesses the difficulty by stating the theorem thus: when a rational or medial area is applied to [i.e. divided by] an irrational line its width will be the cognate irrational. This might be read in either of two ways: that there will be *some* rational or medial area which is the product of the irrationals $c + d$, $c - d$; or that *any* rational or medial area $A = (c \pm d) \cdot x$ gives rise to an irrational line $x$ falling within the cognate *class*. Pappus does indeed intend the latter, as he goes on to prove in §II, 21–23. I will sketch his proof for the additive irrational $c + d$, when A is rational (II, 22). In a lemma (II, 21) Pappus shows geometrically that if $A = (c + d) \cdot x = r^2$ and $(c + d)^2 = r(a + b)$ and $x^2 = r \cdot y$, then $y \cdot (a + b) = r^2$. From *Elements* X, 112 it follows that $y = a' - b'$, an apotome of the same order as the binomial $a + b$; since $x^2 = ry$, one has that $x$ is a subtractive irrational of the same class as the additive irrational $c + d$. In II, 23 Pappus adapts this result for the case of medial area A′. For if $A' = (c + d) \cdot x' = r \cdot r'$, it follows that $x' : x = r' : r$. Thus $x'$ is commensurable in square with line $x$ already determined, and so falls within the same irrational class, namely that cognate to $c + d$. Junge notes that neither Pappus nor Euclid actually proves the theorem needed for this conclusion (p. 150, n 131); it is the same as that discussed in note 38 above. We observe that the present theorem of course applies for the subtractive irrationals $c - d$, although Pappus does not state this result. He needs only the additive cases for effecting his principal theorem on the means: that given any irrational $c + d$, the harmonic mean of its terms $c$, $d$ is a subtractive irrational in the class cognate to $c + d$ (II, 19 f; for $h(c + d) = 2cd$, a rational or medial area). One may note further a corollary to both of Pappus' theorems here, but not stated by him, is that the irrational $c' \mp d'$ derived in either of them in association with the irrational $c \pm d$ is such that $c' : c = d' : d$ (cf. note 38).

[41]Heath [1926, III, 255].

[42]*Ibid.*, p. 246.

[43]*Commentary on Book X*, §II, 17 ff; cf. note 40 above. Note that Pappus' treatment can be considerably abridged via the relations given in note 33 and surely available to him: the harmonic mean will be a line commensurable in square with $c - d$, since $h : c - d = b : (a^2 - b^2)^{1/2}$, a ratio of rational lines. Thus, by the result in note 38 (which Pappus in fact assumes in his own proof in II, 23) $h$ will be an irrational line $c' - d'$ of the same class as $c - d$ and for which $c : c' = d : d'$.

[44]Zeuthen [1896].

[45]An "analysis" deriving the fourth binomial might run thus: assuming $a$, $b$ rational, there are integers $m$, $n$ such that $b^2 : a^2 - b^2 = m : n$. Thus, $b^2 : a^2 = m : m + n$ and $a^2 - b^2 : a^2 = n : m + n$. Since (1) from the definition of the fourth binomial $a$ is commensurable neither with $b$ nor with $(a^2 - b^2)^{1/2}$, it follows that (2) neither of the ratios $m : m + n$, $n : m + n$ can be a ratio of square integers (cf. X, 9). In Euclid's synthesis of the problem of finding the fourth binomial in X, 51 integers $m$, $n$ satisfying conditions (2) are assumed, and from them the lines $a$, $b$ are produced and shown to have the properties (1). It is noteworthy that Euclid does not attempt to justify the assumption of $m$, $n$ satisfying (2) by any explicit procedure or construction. The lapse is not particularly serious, if one takes his effort here to be the reduction of the stated problem to another; but it would surely be striking, if his intent here were to effect a proof of the existence of the fourth binomial.

[46]Proclus [1873, 67].

[47]Pappus [1876-, II, 652–654]; cited by Heath [1921, I, 432 f].

[48]Note that since Book X in one important instance does not establish a result adequate for the purposes of the application in Book XIII (cf. note 38), one cannot well suppose that Book X was specifically directed toward the needs of Book XIII. This observation is developed below.

[49]Pappus [1876, I, 178–186]; cf. Heath [1926, III, 9 f]. Pappus does not indicate the wider geometric context of his two constructions, nor does he identify the geometers responsible for them.

[50]For a survey, see Heath [1926, III, 255–259] and Junge [1930, 26–29].

[51]See the collection of "testimonia" in Euclid [1969-, III, xvi–xxix]. The commentators rarely get beyond definitions and generalities (cf. also Proclus [1873, 60 f]). In addition to the theorems discussed above in notes 40 and 43, Pappus includes the adaptations of Euclid's application theorems X, 60–65, 97–102, where the line of the application is a medial instead of a rational. This yields that if $(c + d)^2$ is applied to a medial line $m$, the resulting width will be a first or second bimedial; similarly, if $(c - d)^2$ is applied to $m$, the result will be one of the two bimedial differences. Pappus' method is as follows: let

$(c \pm d)^2 = m \cdot x = r(a \pm b)$, and set $mc' = ra$, $md' = rb$. Consider first the case where $c \pm d$ are of classes 1 or 4 (Pappus, II, 27 for the additive cases, 30 bimedial difference), but when $m^2 \not\mathrm{C}\, 2cd$, then $b' \not\mathrm{C}\, a \mathrm{C}\, r$, so that $x$ will be a second bimedial (or difference). If next $c \pm d$ is of class 2 or 5 (where $a \not\mathrm{C}\, r$ and $b \mathrm{C}\, r$), we note that $m^2 : c^2 + d^2 = rr' : ra = 2b : b'$. Thus, if $m^2 \mathrm{C}\, c^2 + d^2$, for the subtractive); here $a \mathrm{C}\, r$ and $b \not\mathrm{C}\, r$. Since $mc' = ra$, a rational area, $c'$ will be medial (Pappus, II, 26; cf. *Elements* X, 25). Further, $md' = rb$ is a medial area. Pappus now distinguishes two cases: if $m^2 \mathrm{C}\, md'$, then $m \mathrm{C}\, d'$ so that $d'$ is medial (*Elements* X, 23); he shows also that here $c' \cdot d'$ is rational, so that $c' \pm d'$ is a first bimedial line. When $m^2 \not\mathrm{C}\, md'$, then $d'$ will be a medial line commensurable with $m$ in square only; he shows that $c' \cdot d'$ is here medial, so that $c' \pm d'$ is a second bimedial. Similar treatments follow for the additive irrationals of cases 2 and 5 (II, 28), 3 and 6 (29), and the subtractive classes 2 and 5 (31), 3 and 6 (32).

It is interesting to see how this works out via the alternative "side" representations of the irrationals. As before, we set $(c \pm d)^2 = m \cdot x = r(a \pm b)$; next set $(a \pm b)^2 = r'(a' \pm b')$, where $m^2 = r \cdot r'$. Since $a \pm b$ is binomial (apotome), $a' \pm b'$ will be a first order binomial (apotome) relative to $r'$; that is, $a' \mathrm{C}\, r'$ and $a' \mathrm{C}\, (a'^2 - b'^2)^{1/2}$. Now, $x^2 : r^2 = (a \pm b)^2 : m^2 = (a' \pm b') : r$, so that $x^2 = r(a' \pm b')$. Since $a' \mathrm{C}\, (a'^2 - b'^2)^{1/2}$ and $a' \mathrm{C}\, r' \not\mathrm{C}\, r$ (for $r \cdot r' = m^2$ is medial), $a' \pm b'$ will be a second or third order binomial (apotome) relative to $r$. Thus, $x$ will be a first or second bimedial (or bimedial difference). To distinguish the cases, we consider first $c \pm d$ to be of class 1 or 4 (so that $a \mathrm{C}\, r$ and $b \not\mathrm{C}\, r$); since $m^2 : 2cd = rr' : rb = 2a : b'$ (for $rb = 2cd$, and $r'b' = 2ab$): when $m^2 \mathrm{C}\, 2cd$, then $b' \mathrm{C}\, a \mathrm{C}\, r$, so that $x$ will be a first bimedial (or $b' \mathrm{C}\, b \mathrm{C}\, r$, so that $x$ will be a first bimedial (or difference); but if $m^2 \not\mathrm{C}\, c^2 + d^2$, then $b' \not\mathrm{C}\, b \mathrm{C}\, r$, so that $x$ will be a second bimedial (or difference). For the last cases $c \pm d$ of class 3 or 6 (where $a \not\mathrm{C}\, r$ and $b \not\mathrm{C}\, r$), the pattern partly breaks down (as it must also for Pappus). In view of the relations used in the previous two cases, if either $m^2 \mathrm{C}\, 2cd$ or $m^2 \mathrm{C}\, c^2 + d^2$ (note that it cannot be commensurable with both, since $a \not\mathrm{C}\, b$), then $b' \mathrm{C}\, a \not\mathrm{C}\, r$ in the former case, or $b' \mathrm{C}\, b \not\mathrm{C}\, r$ in the latter; thus $x$ will be a second bimedial (or difference). But if $m^2$ is incommensurable with both $2cd$ and $c^2 + d^2$, we cannot specify in general when $b' \mathrm{C}\, r$ and when $b' \not\mathrm{C}\, r$, so that $x$ may accordingly be either a first or a second bimedial. Note that in all these instances, the conditions determining whether $x$ is a first or a second bimedial (or difference) can be referred back to the form adopted by Pappus via the relation $rb' = 2c'd'$; for in the first bimedial, $b' \mathrm{C}\, r$, whence $c'd'$ is a rational area; when $b' \not\mathrm{C}\, r$, then $c'd'$ is medial (see Table II).

[52] Mueller [1981, 271].

[53] At the time of his course of lectures on the history of the Pell equation (Institute for Advanced Study, 1978–79), Professor A. Weil expressed to me his disappointment over Euclid's utter failure to perceive this potentially fruitful development of the theory of Book X. The meager hints we might glean from Archimedes and Diophantus are little encouragement for supposing that the Greeks recognized these possibilities or pursued them to any length.

[54]Aristotle's theory of the syllogism in *Prior analytics* I and II might be compared with Euclid's Book X as an effort toward the complete systematization of a field. But other mathematical works, both by Euclid and other geometers, rarely even adopt the axiomatic form; and when they do, as in Archimedes' *Plane equilibria* I or the Euclidean *Optics*, the execution is far from complete, whether from the logical or the technical viewpoint.

[55]Is this what van der Waerden has in mind when he praises, "The author succeeded admirably in hiding his line of thought..."? [1963, 172]. Note that he wishes to identify that author as Theaetetus.

[56]Cf. *Commentary* II, 17 and 18 [1930, 138, 143].

## References


Euclid (1969–73), *Elementa*, (J. L. Heiberg (ed.)), rev. E. Stamatis, 4 vols., Leipzig (based on the first Heiberg edition, 4 vols., 1883–85).

Heath, T. L. (1921), *A history of Greek mathematics*, 2 vols., Oxford.

______ (1926), *The thirteen books of Euclid's Elements*, 2nd ed., 3 vols., Cambridge.

______ (1949), *Mathematics in Aristotle*, Oxford.

Junge, see Pappus (1930).

Knorr, W. R. (1975), *The evolution of the Euclidean Elements*, Dordrecht, Netherlands.

______ (1978), *Archimedes and the pre-Euclidean proportion theory*, Arch. Internat. Hist. Sci., **28**, 183–244.

______ (1981), *Aristotle and incommensurability*, Arch. Hist. Exact Sci., **24**, 1–9.

Maracchia, S. (1980), *Aristotele e l'incommensurabilità*, Arch. Hist. Exact Sci., **21**, 201–228.

Mueller, I. (1981), *Philosophy of mathematics and deductive structure in Euclid's Elements*, Cambridge, Mass.

Neugebauer, O. (1957), *The exact sciences in antiquity*, 2nd ed., Providence, R. I.

Pappus (1876–78), *Collectionis quae supersunt* (F. Hultsch (ed.)), 3 vols., Berlin.

______ (1930), *The commentary of Pappus on Book* X *of Euclid's Elements* (text and translation by W. Thomson; notes and introduction by G. Junge), Cambridge, Mass.

Proclus (1873), *In primum Euclidis Elementorum librum commentarii* (G. Friedlein (ed.)), Leipzig.

Thomson, see Pappus (1930).

van der Waerden, B. L. (1963), *Science awakening*, rev. ed., New York.

Zeuthen, H. G. (1896), *Die geometrische Konstruktion als 'Existenzbeweis' in der antiken Geometrie*, Math. Ann., **47**, 222–228.