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ON THE CONSTRUCTION OF INFINITELY MANY, MUTUALLY NONCONGRUENT, EXAMPLES OF MINIMAL IMBEDDINGS OF S^{2n-1} INTO CP^n, $n \geq 2$

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1. Introduction. In the realm of Riemannian manifolds, the family of simply-connected symmetric spaces of compact type (resp. noncompact type) provides a rich family of testing spaces which are natural generalizations of the classical spherical (resp. hyperbolic) space. Therefore, in the study of Riemannian geometry, it is rather natural to seek generalizations of various basic results of the spherical (resp. hyperbolic) geometry to that of compact (resp. noncompact) symmetric spaces. One of the simplest basic facts in spherical geometry is the existence of "*equator*" which is the simplest codimension-one, closed minimal, submanifold in $S^n(1)$. On the other hand, in a recent work of the second author [**5**], infinitely many mutually noncongruent minimal imbedings of S^{n-1} into $S^n(1)$ were constructed for the cases $n = 4, 5, 6, 7, 8, 10, 12$ and 14. The feeling is that such infinite examples should exist for spheres of all dimensions $n \geq 4$. Thus, one is naturally led to the following problems of [**3**, **4**]:

(i) *Problem on the existence of an equator.* Let M^n be a given simply-connected symmetric space of compact type. Does there exist a *minimal, imbedding of* S^{n-1} into M^n?

(ii) *Problem of spherical Bernstein type.* Suppose there is already a *known* example of minimal imbedding of S^{n-1} into M^n. Is it true that *any minimal imbedding* of S^{n-1} into M^n is necessarily congruent to the above known example?

However, in view of the negative answer to the spherical Bernstein problem, it will, in fact, be much more interesting if the answer of the following problem happens to be true in its full generality.

(iii) *Problem of opposite spherical Bernstein type.* Let M^n be a simply-connected symmetric space of compact type, $n \geq 4$. Do there exist *infinitely many* mutually noncongruent examples of *minimal imbeddings of* S^{n-1} into M^n?

The above problem is proved to be affirmative for spheres of dimension $n = 4, 5, 6, 7, 8, 10, 12$ and 14 in [**5**] and for $S^n(1) \times S^n(1)$ for $n = 2, 3$ in [**4**]. We announce here that the above problem is also affirmative for *all* CP^n, $n \geq 2$, namely, the following theorem:

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MAIN THEOREM. *There exist infinitely many mutually noncongruent examples of minimal imbeddings of S^{2n-1} into CP^n, $n \geq 2$.*

2. The construction. In the framework of Riemannian manifolds, the multiplicative group of unit complexes, S^1, acts isometrically on $S^{2n+1}(1) \subseteq C^{n+1}$ and the Riemannian metric on CP^n is exactly the *orbital distance* metric of the S^1-action on $S^{2n+1}(1)$. Since $S^{2n+1}(1) \xrightarrow{\pi} CP^n$ is a Riemannian submersion with totally geodesic fibres, it is easy to see that $\sum^{2n-1}$ is a *minimal* hypersurface in CP^n if and only if $\pi^{-1}(\sum^{2n-1})$ is an *S^1-invariant minimal* hypersurface in $S^{2n+1}(1)$. Therefore, the proof of the above Main Theorem can be reduced to the construction of infinitely many distinct examples of *S^1-equivariant, minimal imbeddings* of $(S^1, S^1 \times S^{2n-1})$ into $(S^1, S^{2n+1}(1))$. The crucial idea makes the construction of such a family of infinitely many examples, at all, possible is based on the specific orbital geometry of the orthogonal action of $G = U(1) \times U(1) \times U(n-1)$ on $S^{2n+1}(1)$. The orbit space, $\Delta = S^{2n+1}/G$, equipped with the orbital distance metric is a spherical triangle. For our purpose, it is convenient to parametrize Δ by a polar coordinate system, namely,

$$\Delta = \{(r, \theta); 0 \leq r, \theta \leq \pi/2, ds^2 = dr^2 + \sin^2 r d\theta^2\}.$$

The generic G-orbits are of the type $S^1 \times S^1 \times S^{2n-3}$ and the volume function, which records the volume of generic orbits, is as follows:

$$v(r, \theta) = C \sin^2 r \cos^{2n-3} r \sin\theta \cos\theta$$

where C is the volume of $S^1(1) \times S^1(1) \times S^{2n-3}(1)$.

Based on the above orbital geometry of $(G, S^{2n+1}(1))$, it is not difficult to derive the following characteristic ODE for the "*generating curves*" of *G-invariant*, minimal hypersurface in $S^{2n+1}(1)$, namely,

$$\frac{d\alpha}{ds} + 3\cos r \frac{d\theta}{ds} - \frac{2}{\sin r} \cot(2\theta) \frac{dr}{ds} - (2n-3) \sin r \tan r \frac{d\theta}{ds} = 0 \tag{1}$$

where α is the angle between the generating curve, γ, and the radial direction $\partial/\partial r$.

Let the three sides of Δ be denoted by $\overline{OA}$, $\overline{OB}$ and $\overline{AB}$ which are respectively given by the conditions $\theta = 0$, $\theta = \pi/2$ and $r = \pi/2$. Observe that the above ODE becomes singular at $\overline{OA}$, $\overline{OB}$ and $\overline{AB}$. However, it follows from a result of [**3**] that there exists a unique solution curve, γ_p, of (1) whose initial point is a given interior point, p, of $\overline{OA}$, $\overline{OB}$ or $\overline{AB}$. We shall denote the unique solution curve of (1) with $p = (a, 0)$ as its initial point simply by γ_a, $0 < a < \pi/2$, and the θ-bisector, i.e. $\theta = \pi/4$, by $\overline{OC}$. Then, the main theorem follows readily from the following more specific results:

THEOREM. *To each positive integer i, there exists a suitable value $0 < a_i \leq \sin^{-1}\sqrt{1/2n}$ such that γ_{a_i} is a solution curve of the ODE (1) without self-intersection which starts at $(a_i, 0)$, intersects with $\overline{OC}$ at exactly i points and ends at an interior point of $\overline{AB}$. The inverse images $\{\Gamma_i = \pi^{-1}(\gamma_{a_i}), i = 1, 2, \ldots\}$ provide an infinite family of G-equivariant, minimal imbeddings of $S^1 \times S^{2n-1}$ into $S^{2n+1}(1)$. The quotients of Γ_i, $\{\sum_i = \Gamma_i/S^1, i = 1, 2 \ldots\}$, provide an infinite family of $\overline{G}$-equivariant, $\overline{G} = G/S^1$, minimal imbeddings of S^{2n-1} into CP^n.*

The proof of the above theorem and some other related results will be published elsewhere.

REFERENCES

1. E. Bombieri, E. DeGiorfi and E. Guisti, *Minimal cones and the Bernstein problem*, Invent. Math. **7** (1969), 243–268.
2. S. S. Chern, *Differential geometry, its past and its future*, Actes Congr. Internat. Math. (Nice, 1970), Tome 1, Gauthier-Villars, Paris, 1971, pp. 41–53.
3. W. T. Hsiang and W. Y. Hsiang, *On the existence of codimension one minimal spheres in compact symmetric spaces of rank* 2. II, J. Differential Geometry **17** (1982), 583–594.
4. ——, *An infinite family of minimal imbeddings of* S^{2n-1} *into* $S^n(1) \times S^n(1), n = 2, 3$. I (to appear).
5. W. Y. Hsiang, *Minimal cones and the spherical Bernstein problem*. I, II (to appear).

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