and beyond this explains at crucial points the reasons for certain steps which could baffle a newcomer to the field.

REFERENCES

1. Felix E. Browder (ed.), *Mathematical developments arising from Hilbert problems*, Proc. Sympos. Pure Math., vol. 28, Amer. Math. Soc., Providence, R. I., 1976.

2. G. Hamel, *Ueber die Geometrien, in denen die Geraden die Kürzesten sind*, Dissertation, Göttingen, 1901, 90 pp.

3. ______, same title, Math. Ann. **57** (1903), 231–264.

HERBERT BUSEMANN

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Transformation groups and representation theory, by Tammo tom Dieck, Lecture Notes in Math., vol. 766, Springer-Verlag, Berlin and New York, 1979, viii + 300 pp., $18.00.

Let G be a topological group and X a topological space. An action of G on X is a continuous map $G \times X \to X$, written $(g, x) \to gx$ on elements, such that $1x = x$ and $g(g'x) = (gg')x$. The study of such group actions is a major and growing branch of topology.

Probably the longest established aspect of this study concerns smooth actions of compact Lie groups on differentiable manifolds. Typically, one tries to classify such actions on a given manifold or to construct particularly nice or particularly pathological examples. A recent concern, still very much in its infancy, is the analysis of the algebraic topology of G-spaces. This book is largely concerned with aspects of this new subject of equivariant homotopy theory.

While some formal theory goes through more generally, it is widely accepted that the appropriate level of generality is to restrict attention to compact Lie groups. Here there is a dichotomy. Many parts of the theory become very much simpler when one restricts further to finite groups, but one feels that one really doesn't understand the theory unless one can carry it out for all compact Lie groups.

The major computable invariants of algebraic topology are "stable". That is, with a shift of indexing, they are the same for a based space X and for its suspensions $\Sigma^n X = X \wedge S^n$. Here the smash product $X \wedge Y$ is the quotient of $X \times Y$ by the wedge, or 1-point union, $X \vee Y$. In equivariant algebraic topology, this description will not do. It makes little sense to restrict attention to spheres with trivial G-action. Since it would be unmanageable to allow spheres with arbitrary G-action, it is best to understand G-spheres to be 1-point compactifications SV of representations V. Here V is a finite-dimensional real inner product space with G acting through isometries. With basepoints fixed under the action of G, "stable" invariants of based G-spaces should be the same for X and for $\Sigma^V X = X \wedge SV$, where G acts diagonally on the smash product. To make sense of this, one must be able to shift indices by representations. Therefore the invariants should be indexed on the real representation ring $RO(G)$. These ideas lead to the notions of $RO(G)$-graded homology and cohomology theories on G-spaces. While such theories are only briefly considered in this book, I wish to emphasize them since it is my belief that the greatest value of the book in the long run will be that it sets up the essential foundations for the calculational study of such theories.

It is a truism that the relation $\pi_n S^n = Z$ for $n \geqslant 1$ is the essential starting point for calculations in algebraic topology. From the point of view of stable invariants, what is important is not this equation but its stable version $\pi_0 QS^0 = Z$, where QX is the union of the spaces $\Omega^n\Sigma^n X$ of continuous based maps $S^n \to \Sigma^n X$. Perhaps the central theme of this book is the analysis of the equivariant version of this equation.

The equivariant version of QX is clear enough. For a based G-space X, consider the space $\Omega^v\Sigma^v X$ of based maps $SV \to \Sigma^v X$. G acts on $\Omega^v\Sigma^v X$ by conjugation, $(gf)(s) = g \cdot f(g^{-1}s)$ for $s \in SV$ and $f\colon SV \to \Sigma^v X$. Note that f is a fixed point of $\Omega^v\Sigma^v X$ if and only if it is a G-map. Let $U = \bigoplus_i V_i^\infty$, where V_i runs through a set of representatives for the irreducible representations of G and V_i^∞ is the sum of countably many copies of V_i. For subrepresentations $V \subset W$ of U, we can suspend a map $SV \to \Sigma^v X$ to a map $SW \to \Sigma^w X$ by smashing with the identity map of the orthogonal complement of V in W. This gives an inclusion of G-spaces $\Omega^v\Sigma^v X \to \Omega^w\Sigma^w X$. Passing to the union over $V \subset U$, we obtain a G-space QX. In particular, we have a G-space $QS^0 = \bigcup \Omega^v SV$. (Note that QX is G-homeomorphic to $\Omega^v Q\Sigma^v X$; $\Sigma^\infty X = \{Q\Sigma^v X\}$ is an example of a G-spectrum [**4**].)

Define $\pi_n^G X = \pi_n(X^G)$, where X^G is the fixed point subspace $\{x | gx = x$ for all $g\}$. In particular, $\pi_0^G QS^0$ may be viewed as the set of homotopy classes of stable G-maps between G-spheres. This set is a ring. Its addition comes from the identification $QS^0 = \Omega QS^1$ and loop addition. Its multiplication comes from composition of G-maps. This ring is every bit as important in equivariant algebraic topology as is the ring $\pi_0 QS^0 = Z$ in ordinary algebraic topology.

What is this ring algebraically? For finite G, it admits a very simple description. Consider the semiring of isomorphism classes of finite G-sets. Its addition and multiplication are given by disjoint union and Cartesian product. Applying the Grothendieck construction (that is, adjoining additive inverses), one obtains a ring $A(G)$ called the Burnside ring of G. It is a basic insight of Segal [**6**] that $A(G)$ is isomorphic to $\pi_0^G QS^0$. (The proof sketched in [**6**] is incorrect, but several people, among them tom Dieck, later supplied correct arguments.) Tom Dieck gives a very clear description of the algebraic properties of $A(G)$ and of its role in proving "induction theorems" in Chapters 1 and 6, this material being mainly due to Dress [**2**]. For any commutative ring k, a finite G-set S determines a k-representation V of G with k-basis the elements of S. Conversely, if k is finite, then a k-representation of G determines a finite G-set by neglect of linear structure. Expanding and clarifying ideas of Segal [**7**], tom Dieck exploits these correspondences to study the relationship between the Burnside ring and representation rings in

Chapters 2 and 4. (Chapter 3, which is a bit of a digression, is an exposition of the work of Atiyah and Tall [1] on λ-rings.)

What is the ring $\pi_0^G QS^0$ for general compact Lie groups G? To answer this question, tom Dieck constructs the appropriate Burnside ring $A(G)$ and analyzes its algebraic properties in Chapter 5. This chapter is the longest and most successful in the book and should be required reading for anyone interested in equivariant homotopy theory. It is based on a series of papers by the author, but it also contains interesting material not previously published. In Chapter 8, just enough foundational material in equivariant homotopy theory is developed to allow a rigorous proof of the fundamental isomorphism $A(G) \cong \pi_0^G QS^0$.

On general nonsense grounds, if E^* is an $RO(G)$-graded cohomology theory on G-spaces, then each $E^v X$ is a module over the Burnside ring. We can therefore localize $E^v X$ at any prime ideal of $A(G)$. Tom Dieck uses this technique among others to prove a series of interesting localization and splitting theorems for the calculation of $RO(G)$-graded homology and cohomology theories in Chapter 7. It is to be emphasized that G is a compact Lie group and not just a finite group throughout this chapter (this being why the induction theorems of the preceding chapter are not used). While this material is not easy reading, it will most certainly prove to be of great value in future work. Tom Dieck uses these results to study equivariant K-theory, and he has used them elsewhere to study equivariant cobordism. Curiously, what one thinks of as the most elementary and fundamental example, namely "ordinary" $RO(G)$-graded cohomology, did not yet exist when the book was written. In fact, such theories were only invented very recently [4], [8]. In my opinion, this development greatly increases the force of this part of tom Dieck's work.

The rest of the book is concerned with the study of when two representations V and W are stably G-equivalent in the sense that $S(V \oplus Z)$ and $S(W \oplus Z)$ are G-homotopy equivalent for some other representation Z, and with the analogous stable G-equivalence problem for G-vector bundles. Most of this material is restricted to finite G. (This is not a serious restriction in the case of representations but is in the case of bundles.)

For finite p-groups, the problem for representations is solved completely. For general finite groups G, the problem is solved under a weaker equivalence relation then stable G-equivalence. For general compact Lie groups G, an attractive conceptual criterion in terms of projective and free $A(G)$-modules for when two given representations are stably G-equivalent is derived. This is used to obtain substantial partial information on the set of stable equivalence classes of representations of G. These results are taken from a series of papers by tom Dieck and tom Dieck and Petrie.

The last chapter, on G-vector bundles, is restricted to finite p-groups and is in more preliminary form than the rest of the book. While it contains some very useful ideas, its results have been overtaken by later developments [3], [5]. (Editorially, a paragraph is missing from the nice proof of the nonequivariant Adams conjecture on p. 285. Mathematically, some statements are left unproven and others are proven sketchily or incorrectly. In particular, the proof by analogy in the first paragraph on p. 292 is unconvincing and the proof in the following paragraph tacitly assumes the converse to the non-equivariant complex Adams conjecture, which is false.)

This volume is addressed to experts in algebraic topology. There is no general introduction and the individual chapters have at most a few sentences of introduction. There is no index and a quite inadequate list of notations. On the other hand, most chapters end with historical comments and a guide to the relevant literature, and there is a very useful bibliography (although several references in the text failed to reach it). The "exercises" tend to be just that early in the book but become references to deeper results and research problems later on. There are numerous misprints. In particular, symbols meant to be completed by hand rather than by typewriter are often incomplete. For example, $\in$ or $=$ may appear where $\notin$ or $\neq$ is intended (e.g., in the statements of Propositions 7.4.3 and 7.7.3). Nevertheless, the experts owe tom Dieck a considerable debt of gratitude, since they will be able to use the book to get some feel for this fascinating new direction in algebraic topology.

REFERENCES

1. M. F. Atiyah and D. O. Tall, *Group representations, λ-rings, and the J-homomorphism*, Topology **8** (1969), 253–297.

2. A. Dress, *Contributions to the theory of induced representations*, Lecture Notes in Math., vol. 342, Springer-Verlag, Berlin and New York, 1973, pp. 183–240.

3. H. Hauschild and S. Waner, *The equivariant Dold theorem* mod k (to appear).

4. G. Lewis, J. P. May and J. McClure, *Ordinary $RO(G)$-graded cohomology* (to appear).

5. J. McClure, *The groups $JO_G(X)$* (to appear).

6. G. B. Segal, *Equivariant stable homotopy theory*, Actes Internat. Congres Math., Tome 2, 1970, pp. 59–63.

7. ______, *Permutation representations of finite p-groups*, Quart. J. Math. Oxford (2) **23** (1972), 375–381.

8. S. Waner, *Equivariant $RO(G)$-graded singular cohomology* (preprint).

J. P. MAY

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Integral representations, by Irving Reiner and Klaus W. Roggenkamp, Lecture Notes in Math., vol. 744, Springer-Verlag, Berlin, Heidelberg, 1979, 272 pp., $14.30.

Representations of a finite group G are finitely generated RG-modules, where R is a commutative ring. Thus representation theory is largely concerned with the commutative monoids $m(RG)$ where, for any ring Λ, $m(\Lambda)$ denotes the monoid of isomorphism classes of finitely generated Λ-modules with addition given by the direct sum.

Classically R is taken to be the complex numbers. The monoids $m(\mathbf{C}G)$ have a very simple description: they are freely generated by finitely many irreducible modules. Indeed for any field K whose characteristic does not