Geometric asymptotics, by Victor Guillemin and Shlomo Sternberg, Math. Surveys, no. 14, Amer. Math. Soc., Providence, R.I., 1977, xviii + 474 pp., $34.40.

Symplectic geometry and Fourier analysis, by Nolan R. Wallach, Math Sci. Press, Brookline, Massachusetts, 1977, xvii + 436 pp.

1. Introduction. The use of symplectic geometry to describe classical mechanics and to understand it on a deeper level has its origins in the work of Poincaré (1889), Cartan (1922), Siegel (1950) and Reeb (1951). By the 1960s this topic was widely known and was available from several sources such as Mackey [1963], Sternberg [1964], Abraham [1967], Hermann [1968] and Godbillon [1969].

During the 1960s a new direction and impetus to the field arose when deep links between symplectic geometry, group representations, quantization and linear partial differential equations were found by Keller [1958], Segal [1960], [1965], Kirillov [1962], Maslov [1965], Egorov [1969], Kostant [1970], Souriau [1970], Hörmander [1971] and Duistermaat and Hörmander [1972], to mention some of the key contributors. The subject is evolving rapidly and therefore a definitive treatise is not possible at the present time. Nevertheless, the books under review attempt to describe some of these new links.

To penetrate to the basic ideas in either of the books requires extensive background preparation and a large investment of time. However, some of the key ideas are already present in the simplest examples. Therefore we shall spend some time discussing the one dimensional Schrödinger equation and the relation between classical and quantum mechanics. This will give the potential reader a glimpse at the theory and what type of results are obtained.

2. The one-dimensional Schrödinger equation. Let $V: \mathbf{R} \to \mathbf{R}$ be the potential, $\psi: \mathbf{R} \to \mathbf{C}$ the wave function, and let E, $\hbar$, m be constants (energy, Planck's constant and mass, respectively). Consider the stationary Schrödinger equation:

$$L\psi = E\psi$$

where

$$L\psi = -\frac{\hbar^2}{2m} \psi'' + V\psi \qquad \text{(S)}$$

and the Hamilton-Jacobi equation for Hamilton's principal function S: $\mathbf{R} \to \mathbf{R}$:

$$\frac{1}{2m}(S')^2 + V = E. \qquad \text{(H-J)}$$

The Hamilton-Jacobi equation is related to Hamilton's equations

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x} \qquad \text{(H)}$$

where $H(x, p) = p^2/2m + V(x)$, as follows. Let $\dot{x} = p/m = \partial H/\partial p$ and let

$p = S'(x)$. Then (H-J) holds for S iff (H) holds for (x, p).

Two related central questions are:

1. THE QUANTIZATION PROBLEM. How does one pass from classical objects to quantum objects? Here, 'objects' can refer to the equations themselves, to solutions, or to properties of the equations or solutions.

2. THE CLASSICAL LIMIT. In what sense are solutions of the Hamilton-Jacobi equation a limit of solutions of the Schrödinger equation as $\hbar \to 0$?

Progress with these questions was made with the basic work of Weyl [1931], Birkhoff [1931], Van Hove [1951], Keller, Maslov, Souriau and Kostant. Van Hove showed that there is no general quantization having all the properties one would want. Van Hove also found some positive results that were rediscovered and extended by Souriau and Kostant in a procedure now called pre-quantization. In studying problem 2 using the WKB method, Keller and Maslov discovered the topological meaning of the corrected Bohr-Sommerfeld quantization rules. The invariant they discovered is now called the *Keller-Maslov-Arnold-Hörmander* index. (Arnold's article [1967] was instrumental in explaining Maslov's ideas to mathematicians.) Our one-dimensional example will contain many of the features of the general case, in terms understandable without a lot of preliminaries.

If S is a solution of (H-J), we try to solve (S) with

$$\psi = e^{iS/\hbar}. \tag{1}$$

Substitution of (1) in (S) gives

$$E\psi = L\psi + \frac{i\hbar}{2m} \psi \frac{\partial^2 S}{\partial x^2}. \tag{2}$$

Equation (2) differs from (S) by a term of order $\hbar$. Next, try

$$\psi = ae^{iS/\hbar}.$$

This time, if S satisfies (H-J) and a satisfies the *transport equation*

$$2a'S' + aS'' = 0 \tag{3}$$

(whose solution is $a = (\text{const.})/\sqrt{|S'|}$) then

$$E\psi = L\psi - \frac{\hbar^2}{2m} \frac{a''}{a} \psi \tag{4}$$

which differs from (S) by a term of order $\hbar^2$. This procedure is usually called the *WKB method* (after G. Wentzel, H. A. Kramers and L. Brillouin, although it probably goes back to Liouville, Green and Lord Rayleigh). One may continue by writing an asymptotic series

$$\psi \sim \sum_{k=0}^{\infty} a_k \hbar^k e^{iS/\hbar}$$

and deriving higher transport equations.

Suppose the energy surface for the classical system has the form shown in Figure 1. There correspond two solutions of (H-J):
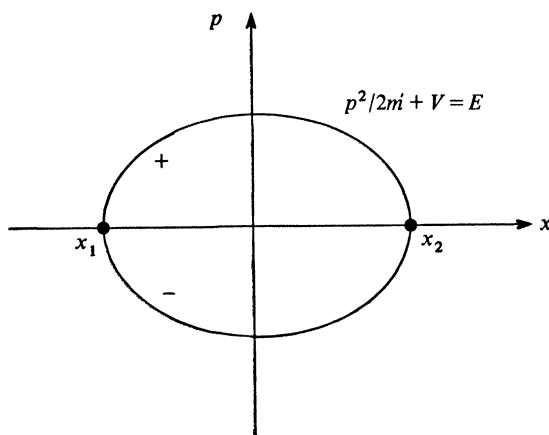
$$p^2/2m + V = E$$

FIGURE 1

$$S = \pm \int p(x)\, dx + C_\pm \tag{5a}$$

where $p(x) = \sqrt{2m(E - V)}$ and $C_\pm$ are constants. From (3) we have corresponding amplitudes

$$a_\pm = \frac{d_\pm}{\left[2m(E - V(x))\right]^{1/4}} \tag{5b}$$

which diverge at x_1 and x_2 and become imaginary outside the interval $[x_1, x_2]$.

The subtlety of questions 1 and 2 centers on the multiple valuedness of S and the presence of the turning points at x_1 and x_2. To get around these difficulties there have been several approaches.

1. Use analytic continuation methods to avoid the turning points. This approach was developed by Zwaan (see Kemble [1937]).

2. Approximate the potential by a linear one near each turning point. Schrödinger's equation then yields an Airy function which is asymptotically matched by Bessel functions (Langer and Jeffreys).

3. Use a modified WKB method near the turning point and an asymptotic expansion (Maslov). We shall describe this method shortly.

There are other approaches too. For instance, Miller and Good [1953] effectively used area preserving maps to deform Figure 1 into that for a harmonic oscillator. The same idea was used by Maslov [1965] for higher order approximations.

To deal with the behavior near x_1 and x_2, we replace $\psi = ae^{iS/\hbar}$ by a superposition of such expressions, i.e. by

$$\psi(x) = \int_{-\infty}^{\infty} a(x, \alpha) e^{iS(x,\alpha)/\hbar}\, d\alpha.$$

(This integral is called an oscillatory function; the theory of such integrals parallels that of Fourier integral operators.)

To simplify ψ slightly, we consider

$$\psi(x) = \int_{-\infty}^{\infty} a(x, \alpha)e^{i(\alpha x - T(\alpha))/\hbar} \, d\alpha. \tag{6}$$

Then

$$L\psi - E\psi = \int_{-\infty}^{\infty} \left[a\left(\frac{\alpha^2}{2m} + V - E \right) + \left(\frac{i\hbar}{2m} \frac{\partial a}{\partial x} - \frac{\hbar^2}{2m} \frac{\partial^2 a}{\partial x^2} \right) \right]$$
$$\times e^{i[\alpha x - T(\alpha)]/\hbar} \, d\alpha. \tag{7}$$

The classical method of stationary phase states that

$$\int_{-\infty}^{\infty} c(x, \alpha)e^{if(x,\alpha)/\hbar} \, d\alpha = \sqrt{2\pi\hbar} \sum_{f_\alpha = 0} \exp\left(\frac{i\pi}{4} \operatorname{sgn} f_{\alpha\alpha} \right) \frac{ce^{if/\hbar}}{f_{\alpha\alpha}} + O(\hbar) \tag{8}$$

where the sum is over all α such that $f_\alpha = \partial f / \partial \alpha$ vanishes; these critical points are assumed to be nondegenerate, i.e. $f_{\alpha\alpha} = \partial^2 f / \partial \alpha^2 \neq 0$. The equation (8) may be found in virtually any text on asymptotic expansions and most books on complex variables. Guillemin and Sternberg's book contains a nice proof of (8).

Applying (8) to (7) and requiring $L\psi - E\psi = O(\hbar)$ gives the condition

$$\frac{\alpha^2}{2m} + V(x) = E \quad \text{whenever} \quad x = -T'(\alpha) \tag{9}$$

i.e. the graph of $-T'$ as a function of p is contained in the energy surface. Here we have the Hamilton-Jacobi equation with the roles of x and p reversed, which is indeed appropriate near the turning points x_1 and x_2.

If we apply (8) to the formula (6) we get

$$\psi(x) = \sqrt{2\pi\hbar} \sum_{x = -T'(p)} \frac{e^{-i\pi\operatorname{sgn} T''(p)/4} e^{i(px + T(p))/\hbar} a(x, p)}{\sqrt{T''(p)}} + O(\hbar) \tag{10}$$

so $\psi \sim \hbar^{1/2}$ and $L\psi - E\psi \sim \hbar^{3/2}$ if (9) holds (with $\alpha = p$).

We now seek to represent ψ near x_1 and x_2 using functions T_1 and T_2 by equations (9) and (10) and seek to represent ψ on the $\pm$ portions by using equations (2) and (5). We are, in effect using a superposition of two WKB approximations.

Notice that

$$\frac{d}{dx}(px + T(p)) = p + \frac{dp}{dx} x + T'(p) \frac{dp}{dx} = p$$

so both S and $px + T(p)$ are given by integrating p with respect to x; i.e., they are both *actions*.

Since $x = -T'(p)$ along the energy curve in Figure 1, we see that $T''(p) > 0$ on the $+$ side and $T''(p) < 0$ on the $-$ side. Thus the term

$$e^{-i\pi\operatorname{sgn} T''(p)/4} \tag{11}$$

in (10) jumps, or suffers a phase shift, as p crosses the x-axis.

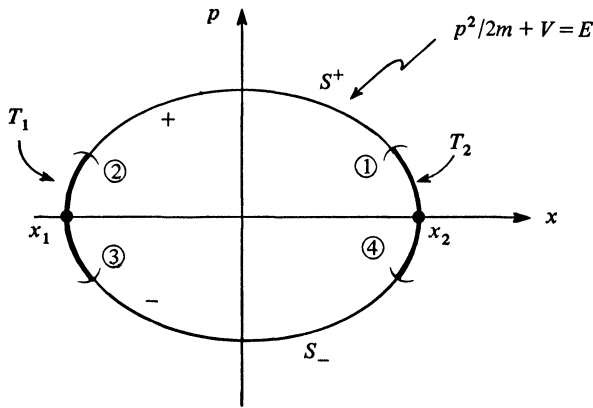In Figure 2 we show the different regions and functions being considered.

FIGURE 2

Observe that the term $\sqrt{T''(p)}$ in (10) is the same as the term $[2m(E - V(x))]^{1/4}$ occurring in (5b). If we absorb the phase shift (11) and $\sqrt{2\pi\hbar}\ a(x, p)$ of (10) into $d_{\pm}$, the solutions will match, except for higher order terms. However, there is an obvious consistency condition; when we circumnavigate the energy curve, we must end up where we started. In fact, matching at points ① ② and ③ fixes all the constants, and ④ will match up only if the phases in (11) match. The phase changes in the exponentials

$$e^{iS/\hbar} \quad \text{and} \quad e^{i(px - T(p))/\hbar}$$

are given by

$$\frac{1}{\hbar} \oint p\, dx$$

since both S and $px - T(p)$ are given by integrating p, as we have said, where $\oint$ is the line integral over the energy curve. (In hamiltonian mechanics, $p\, dx$ is the canonical one form and its differential $dp \wedge dx$ is the symplectic form.) The phase change due to the term (11) is

$$2 \times \left[\frac{\pi}{4} - \left(-\frac{\pi}{4} \right) \right] = \pi,$$

so the consistency condition is

$$\frac{1}{\hbar} \oint p\, dx - \pi = 2\pi n$$

i.e.

$$\oint \frac{p\, dx}{2\pi\hbar} = n + \frac{1}{2}. \tag{12}$$

The $1/2$ is the correction to the Bohr-Sommerfeld rules which one sees, for example, in the harmonic oscillator solution. Equation (12) is the *quantization condition*.

The generalization of (12) reads

$$\frac{1}{2\pi\hbar} \oint_\gamma p_i dq^i - \frac{1}{4} I_\gamma = \text{integer} \tag{13}$$

where I_γ is the *Keller-Maslov-Arnold-Hörmander index* of a closed curve γ. This topological invariant is thus arrived at via the WKB method. To properly understand it in higher dimensions requires a lengthy excursion into the theory of lagrangian submanifolds. However, our simplified example shows that starting with a study of the asymptotic limit $\hbar \to 0$, one is led to quantization conditions; i.e. questions 1 and 2 with which we began this section are intimately related.

3. Dictionary. The overall aims of quantization and geometric asymptotics become clearer if one has in mind some of the classical-quantum correspondences. To this end, we present the table below. (See Sławianowski **[1971]**.)

The basic classical object is a symplectic manifold (T^*X, ω) and the quantum object is the intrinsic Hilbert space $\mathcal{H} = L_2(X)$ of half densities on X. The dictionary sets up a correspondence between operations on each.

	Classical Mechanics	Quantum Mechanics
(a)	immersed lagrangian submanifold $\Lambda \hookrightarrow (T^*X, \omega)$	element $\psi \in L_2(X)$ or $\mathcal{D}'(X)$
(b)	Λ = graph of dS	$\psi = e^{iS/\hbar}$
(c)	multiplication by (-1) on fibers	complex conjugation
(d)	$(T^*X, -\omega)$	dual space
(e)	cartesian product	tensor product
(f)	disjoint union	direct product
(g)	lagrangian submanifold $\Lambda \subset (T^*X, \omega_X) \times (T^*Y, -\omega_Y)$	(possibly unbounded) operator from $L_2(Y)$ to $L_2(X)$
(h)	composition of canonical relations	composition of operators
(i)	graphs of canonical transformations	unitary operators
(j)	Hamilton-Jacobi equation	Schrödinger equation
(k)	coisotropic submanifold $Q \subset T^*X$	involutive system of linear differential equations
(l)	reduced space $Q/Q^\perp$	solution space
(m)	reduction of lagrangian submanifolds	projection onto solution space
(n)	symplectic action (hamiltonian G-space)	unitary representation
(o)	coadjoint orbits (homogeneous hamiltonian G-spaces)	irreducible representations
(p)	reduction of phase space by a symmetry group	multiplicities of irreducibles occuring in a given representation
(q)	momentum mapping	associated representation of the group algebra
(r)	polarization	complete set of observables
(s)	special symplectic structure	representation of a complete set of observables
(t)	change of special symplectic structure (Legendre transformation in the sense of Tulczyjew **[1977]**)	Fourier integral operator

4. The two books. Both of the books under review discuss quantization in detail, although the emphasis is different in each case: Guillemin and Sternberg concentrate on differential equations, Wallach on representation theory.

Both books use the formalism of metaplectic structures and half-forms developed by Blattner, Kostant, and Sternberg. In fact one of the unique features of *Geometric Asymptotics* is the use of half-forms to replace half-densities and the Maslov bundle in the theory of Fourier integral operators and oscillatory functions. This approach, which represents original work of the authors (and which is not published elsewhere) requires extensive algebraic preparation, but it does simplify many calculations.

Guillemin and Sternberg present much more that is new. A Radon-transform approach to the wavefront set was earlier described only briefly in a note by Guillemin and D. Schaeffer. In the chapter entitled *Geometric aspects of distributions*, the authors make clear the elementary nature of some trace formulas which are often considered to be rather deep. There is also a nice discussion of the Plancherel formula for certain noncompact semisimple Lie groups.

In keeping with the teaching purpose of this "Mathematical Survey," Guillemin and Sternberg have included substantial introductory material on optics in order to motivate the subsequent theoretical development. Unfortunately, the exposition here (and sometimes elsewhere in the book) often defeats the authors' purpose by being extremely difficult to follow. The difficulty is compounded by the presence of numerous minor inaccuracies[1] throughout the book.

For example, in the treatment of the Sommerfeld radiation conditions on p. 12, the order estimates for integrals over spheres should be $o(1)$ and $o(R^{-1})$, rather than $o(R^{-2})$ and $o(R^{-3})$ as printed. (This was pointed out to us by Sternberg in a telephone conversation.) But even with this correction made, the reader must be very careful to distinguish the term "eventually outgoing" on p. 12 from the less precise "outgoing wave" on p. 2. In fact, the wave $w_k(t, r)$ on p. 2 is not eventually zero (since it does not satisfy the wave equation at the origin of $\mathbf{R}^3$, where there is a source). One might expect the expert reader to be on the lookout for such distinctions, but they make the going extremely rough for newcomers to the subject (as, for example, the students in a graduate course taught by one of the reviewers using *Geometric Asymptotics*).

Despite the difficulties, we still recommend the book very highly for its wealth of content, with the suggestion that the reader use it in conjunction with other sources, such as Duistermaat **[1974]**, Leray **[1978]**, the original papers, and the book of Wallach, to which we now turn.

The main goal of *Symplectic geometry and Fourier analysis* is a presentation of the Kirillov theory of unitary representations of nilpotent Lie groups, which led to the Auslander-Kostant theory for solvable groups. Closely related to quantization, this is one of the most spectacular applications of symplectic geometry; Wallach brings this out very clearly. Kostant's notes

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[1]Some of these, we understand, were due to difficulties in type-setting and production which were beyond the authors' control.

[1970] have been the standard reference for this subject, but Wallach's exposition is certainly easier going. During the preparations for the Kirillov theory, Wallach's book overlaps that of Guillemin and Sternberg on the topics of "homogeneous symplectic manifolds," "the metaplectic representation" and "quantization." In each case Wallach is less thorough, but more readable.

An appendix on quantum mechanics by Robert Hermann gives an overview of quantum mechanics as a separate discipline from classical mechanics, emphasizing the symplectic viewpoint and its connections (both historical and mathematical) with representation theory. This appendix is informative and is enlivened by its author's usual addition of personal and historical comments, but it does have some deficiencies. For example, no answer to the stated question "why a Hilbert space?" is given; i.e., Gleason's theorem is not discussed. Also, the author speaks as though the symplectic approach to quantum mechanics as an infinite dimensional hamiltonian system has not been developed. (As far as we know, this first occurs in Segal [1965], Marsden [1968] and Chernoff and Marsden [1974].)

5. Outlook. The books under review are certainly valuable additions in the effort to expand our knowledge about symplectic geometry and its applications. However, we can already see research (much of it due to the authors themselves) quickly outrunning these books.

There must be more one can do with Fourier integral operators. From Guillemin and Sternberg's book and Duistermaat [1974], clear relationships with caustics and the elementary singularities (catastrophe theory) are brought out. These ought to have extensions to bifurcation theory in general. Fourier integral operators and the machinery surrounding them have also barely begun to make a dent in other applied areas, such as high frequency gravitational waves and shocks.

The reader should be aware of the applications of Fourier integral operators to classical partial differential operators in addition to those discussed in the books under review and in Hörmander [1971]. At the hands of people like Melrose, Taylor, Ralston and Majda, deep results on the expansion of solutions of the classical wave equation near a caustic have been obtained. We refer to Taylor's excellent book [1979] for details.

REFERENCES

R. Abraham [1967], *Foundations of mechanics*, Benjamin, New York.

R. Abraham and J. Marsden [1978], *Foundations of mechanics*, 2nd ed., Addison-Wesley, Reading, Mass.

V. Arnold [1967], *Characteristic class entering in conditions of quantization*, Functional Anal. Appl. 1, 1–13.

_____, [1978], *Mathematical methods of classical mechanics*, Graduate Texts in Mathematics, vol. 60, Springer-Verlag, Berlin and New York.

G. D. Birkhoff [1931], *Quantum mechanics and asymptotic series*. I, Bull. Amer. Math. Soc. 39, 681–700.

P. R. Chernoff and J. E. Marsden [1974], *Properties of Infinite Dimensional Hamiltonian Systems*, Lecture Notes in Math., vol. 425, Springer-Verlag, New York.

J. J. Duistermaat [1974], *Oscillatory integrals, Lagrange immersions and the unfolding of singularities*, Comm. Pure Appl. Math. 27, 207–281.

J. J. Duistermaat and L. Hörmander [1972], *Fourier integral operators*. II, Acta. Math. **128**, 183–269.

Yu. V. Egorov [1969], *On canonical transformations of pseudo-differential operators*, Uspehi Mat. Nauk **25**, 235–236.

G. Godbillon [1969], *Geometrie differentielle et mécanique analytique*, Hermann, Paris.

R. Hermann [1968], *Differential geometry and the calculus of variations*, Academic Press, New York (Second edition, [1977], Math. Sci. Press).

L. Hörmander [1971], *Fourier integral operators*. I, Acta. Math. **127**, 79–183.

J. B. Keller [1958], *Corrected Bohr-Sommerfeld quantum conditions for nonseparable systems*, Ann. of Physics **4**, 180–188.

E. C. Kemble [1937], *The fundamental principles of quantum mechanics*, McGraw-Hill, New York.

A. A. Kirillov [1962], *Unitary representations of nilpotent Lie groups*, Russian Math. Surveys **17**, 53–104.

B. Kostant [1970], *Quantization and unitary representations*, Lecture Notes in Math., vol. 170, Springer-Verlag, Berlin and New York, pp. 87–208.

J. Leray [1978], *Analyse lagrangienne et mécanique quantique*, R.C.P. 25, vol. 25, I.R.M.A. Strasbourg.

G. W. Mackey [1963], *The mathematical foundations of quantum mechanics*, Benjamin, New York.

J. E. Marsden [1968], *Hamiltonian one parameter groups*, Arch. Rational Mech. Anal. **28**, 362–396.

J. Marsden and A. Weinstein [1974], *Reduction of symplectic manifolds with symmetry*, Rep. Mathematical Phys. **5**, 121–130.

V. P. Maslov [1965], *Theory of perturbations and asymptotic methods*, Moscow State Univ. (French translation, Dunod, 1972).

V. P. Maslov and M. V. Fedorjuk [1976], *Quasiclassical approximation for the equations of quantum mechanics*, Izdat. "Nauka", Moscow (Russian)

S. C. Miller, Jr. and R. H. Good, Jr. [1953], *A WKB-type approximation to the Schrödinger equation*, Phys. Rev. **91**, 174–179.

I. Segal [1960], *Quantization of non-linear systems*, J. Math. Phys. **1**, 468–488.

———, [1965], *Differential operators in the manifold of solutions of a nonlinear differential equation*, J. Math. Pures Appl. **44**, 71–113.

J. J. Słavailable [1971], *Quantum relations remaining valid on the classical level*, Rep. Mathematical Phys. **2**, 11–34.

J. M. Souriau [1970], *Structure des systemes dynamiques*, Dunod, Paris (2nd ed. in preparation).

S. Sternberg [1964], *Lectures on differential geometry*, Prentice-Hall, Englewood Cliffs, N. J.

M. Taylor [1979], *Pseudo differential operators*, (to appear).

W. Tulczyjew [1977], *The Legendre transformation*, Ann. Inst. H. Poincaré, **27**, 101–114.

L. Van Hove [1951], *Sur certaines représentations unitaires d'un groupe infini de transformations*, Acad. Roy. Belg. Cl. Sci. Mem. Coll. in-8° **26**, pp. 61–102.

A. Weinstein [1977], *Lectures on symplectic manifolds*, CBMS Regional Conf. Ser. in Math., no. 29, Amer. Math. Soc., Providence, R. I.

H. Weyl [1931], *The theory of groups and quantum mechanics*, Dover, New York.

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Algebraic geometry, by Robin Hartshorne, Graduate Texts in Mathematics 52, Springer-Verlag, New York, Heidelberg, Berlin, 1977, xvi + 496 pp., $24.50.

After its inception as part of Bernhard Riemann's new function theory, Algebraic Geometry quickly became a central area of nineteenth century