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CONFIGURATION SPACES: APPLICATIONS TO GELFAND-FUKS COHOMOLOGY

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Let M be a manifold and define $F(M, k)$ as the subspace of M^k given by $\{(x_1, \ldots, x_k)|x_i \neq x_j \text{ if } i \neq j\}$. Permuting the coordinates gives a free action of Σ_k, the symmetric group on k letters on $F(M, k)$. If X is a based space, $X^{[k]} = X \wedge \cdots \wedge X$ supports a Σ_k action and we can form

$$B(M, X, k) = F(M, k) \times_{\Sigma_k} X^{[k]}/F(M, k) \times *.$$

The cohomologies of $F(M, k)$ and $B(M, X, k)$ have ubiquitous applications. $H^*(B(M, X, k))$ can be used to evaluate the E_2 term of a spectral sequence converging to the Gelfand-Fuks cohomology of M, [7] or [8]. It can also be used to evaluate the E_2 term of a spectral sequence due to P. Trauber [12] and D. W. Anderson [1] converging to the cohomology of the space of based maps from M to X. The calculations for the case $M = R^m$ give a complete and useful theory of homology operations for m-fold loop spaces [5].

In [4] and [5], the first author has obtained complete information on $H^*(F(R^m, k))$ and $H^*(B(R^m, X, k))$ in conjuction with his work on m-fold loop spaces. In this paper we give some calculations for some other manifolds M. We are most successful with $M^m = R^n \times V$ and with $M = S^m$.

Recall that by [4], $H^*F(R^m, k)$ is generated as an algebra by elements A_{ij} of degree $m - 1$ with $k \geqslant i > j \geqslant 1$ subject to the relations $A_{ir}A_{is} = A_{sr}(A_{is} - A_{ir})$ if $r \leqslant s$. With $A_{ji} = (-1)^m A_{ij}$ for $i > j$, the action of Σ_k is given by $\sigma^* A_{ij} = A_{\sigma i, \sigma j}$.

THEOREM 1. *If V is connected, if $M^m = R^n \times V$ with $n \geqslant 2$, and if all coefficients are in some field, $H^*(F(m, k))$ is isomorphic as an algebra to*

$$H^*(F(R^m, k)) \otimes H^*(V^k)/I$$

where I is the two-sided ideal generated by the elements

$$A_{ij} \otimes (1^{i-1} \times y \times 1^{k-i} - 1^{j-1} \times y \times 1^{k-j})$$

for all i and j and $y \in H^*(V)$. Both $H^*(F(R^m, k))$ and $H^*(V^k)$ are Σ_k algebras

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and the epimorphism from their tensor product to $H^*(F(M, k))$ is a Σ_k algebra morphism.

REMARK. In case V is a point and $n = 2$, Theorem 1 is a result of V. I. Arnold [2] and E. Brieskorn [3] who used different methods than we do. They also did not determine the Σ_k action.

Let L_m^c be the Lie algebra of compactly supported C^∞ vector fields on M.

REMARK. Knowing $H^*(F(M, k))$, the calculation of $H^*(B(M, X, k))$ can be done using the spectral sequence of a cover [10]. If the field has characteristic prime to $k!$ the spectral sequence collapses. Combining Theorem 1 with the Gelfand-Fuks spectral sequence [9] we get

COROLLARY 2. *Let M^m and N^n be two manifolds whose rational Pontrjagin classes vanish and for which $\beta_i(M) = \beta_i(N)$: β_i is the ith Betti number. Then $H^*(L^c_{R^r \times M}) \cong H^*(L^c_{R^s \times N})$ as vector spaces when $r + m = n + s$, $r, s \geqslant 2$.*

PROOF. Theorem 1 assures us that the E_2 terms of the Gelfand-Fuks spectral sequence [9] are equal, and Guillemin [8] and Trauber [12] assure us that the spectral sequences collapse.

Let us turn to the case $M = S^m$. $F(S^1, k)$ is homeomorphic to $S^1 \times F(R^1, k-1)$ and $F(R^1, k-1)$ has the homotopy type of $(k-1)!$ discrete points. In case $m \geqslant 1$, we have

THEOREM 3. *Suppose the coefficient field has characteristic not 2. Then $H^*(F(S^m, k))$ as an algebra over Σ_k is isomorphic to $\Lambda[x] \otimes A_m$ where A_m is the image of $H^*(F(S^m, k))$ in $H^*(F(R^m, k))$ under any embedding $R^m \subset S^m$ and $\Lambda[x]$ is an exterior algebra on a generator x of degree m if m is odd or $2m - 1$ if m is even. Σ_k acts on A_m since A_m is an invariant subgroup of $H^*(F(R^m, k))$. Σ_k fixes x and acts on $\Lambda[x] \otimes A_m$ diagonally.*

Whenever $H_*B(M, X, k)$ is known, E_2 of the Gelfand-Fuks spectral sequence is known by specializing X to be a certain wedge of spheres. Theorem 1 (together with minor modifications in case $n = 1$) yields a complete description of $H_*(B(R^n \times V, X, k); Q)$. The results obtained for Gelfand-Fuks cohomology coincide with those obtained by A. Haefliger [13] who used completely different methods.

COROLLARY 4. *$H^* L^c_{R^n \times V}$ is isomorphic to a free commutative algebra whose generators are explicitly given in terms of $H_* V$ and the dimension of V provided the rational Pontrjagin classes of V vanish and $n \geqslant 1$.*

COROLLARY 5. *$\widetilde{H}^* L_{S^m}$ is additively isomorphic to $\Lambda[x] \otimes B_m$ where $\Lambda[x]$ is as in Theorem 3 and B_m is a certain subspace (but not a subalgebra) of $H^* L^c_{R^m}$.*

Details, further applications, and more extensive computations will appear elsewhere.

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