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TENSOR PRODUCTS OF UNITARY REPRESENTATIONS OF $SL_2(R)$

BY JOE REPKA

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1. **Introduction.** We consider the tensor product of two irreducible unitary representations of $G = SL_2(\mathbf{R})$; in particular, we obtain its reduction as a direct integral of irreducible representations. This question has been solved in certain cases by Pukanszky [4] and Martin [3]. We restate their results and also do the remaining cases.

2. **Notation.** Let $M = \{\pm I\}$; $K = SO_2(\mathbf{R})$; and let A (resp. N) be the subgroup consisting of all positive diagonal matrices (resp. upper triangular unipotent matrices). Let

$$h_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \in A, \qquad k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in K.$$

For $s \in i\mathbf{R}$, $\epsilon \in \hat{M}$, let $\eta_{s,\epsilon}$ be the one-dimensional representation of MAN given by $\eta_{s,\epsilon}\colon mh_tn \longmapsto \epsilon(m) \cdot e^{st}$, $m \in M$, $n \in N$. Let $\pi_{s,\epsilon} = \mathrm{Ind}_{MAN}^G \eta_{s,\epsilon}$, a principal series representation.

For $-1 < \sigma < 0$, let π_σ^c be the (unitary) complementary series representation which is infinitesimally isomorphic to the "nonunitary principal series" representation induced from the representation of MAN given by $mh_tn \longmapsto \exp(\sigma t)$.

The representations π_σ^c are all irreducible, as are all the $\pi_{s,\epsilon}$, except when $s = 0$ and $\epsilon \in \hat{M}$ is nontrivial. In this case, $\pi_{0,\epsilon}$ is the direct sum of two irreducible representations, denoted $\pi_{0,\epsilon}^+$ and $\pi_{0,\epsilon}^-$.

For $n \in \mathbf{Z}$, define $\chi_n \in \hat{K}$ by $\chi_n(k_\theta) = e^{in\theta}$. For $n \geqslant 2$, we let T_n (resp. T_{-n}) be the discrete series representation with lowest weight n (resp. highest weight $-n$). We also let $T_1 = \pi_{0,\epsilon}^+$, $T_{-1} = \pi_{0,\epsilon}^-$, the so-called "mock discrete series representations" with extreme weights 1 and -1 respectively.

The representations we have described exhaust the irreducible unitary representations of G. For details, see, e.g., Lang [2].

3. **A preliminary result.** Before proceeding, we state a very easy but useful fact, for any separable locally compact group.

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PROPOSITION. *Let π_1, π_2 be irreducible unitary representations of G which are of class L^p and L^q respectively. Suppose $1/p + 1/q \geqslant 1/2$. Then $\pi_1 \otimes \pi_2$ is quasi-contained in the regular representation.*

4. Reduction of tensor products. We have the following results, of which (a), (b) and (c) are due to Pukanszky [4] in the case of even parity, and (a) is due to Martin [3] in the remaining cases.

Let $r, s \in i\mathbf{R}$; $-1 < \sigma, \tau < 0$; $\epsilon, \epsilon' \in \hat{M}$. Let $\epsilon'' \in \hat{M}$ be the trivial character. Let $m, n \geqslant 1$ be integers. Let $\epsilon(m) \in \hat{M}$ be trivial or not according as m is even or odd. Then

(a) $$\pi_{r,\epsilon} \otimes \pi_{s,\epsilon'} \approx 2\left(\int_{\mathbf{R}^+} \pi_{it,\,\epsilon\epsilon'}\, dt\right) \oplus \left(\bigoplus_{|k|\geqslant 2;k\equiv\epsilon\epsilon'} T_k\right).$$

(Here the notation $k \equiv \epsilon\epsilon'$ means the summation is over all even k if $\epsilon\epsilon'$ is trivial and over all odd k otherwise.)

(b) $$\pi_{r,\epsilon} \otimes \pi^c_\sigma \approx 2\left(\int_{\mathbf{R}^+} \pi_{it,\epsilon}\, dt\right) \oplus \left(\bigoplus_{|k|\geqslant 2;k\equiv\epsilon} T_k\right).$$

(c) For $\sigma + \tau \geqslant -1$,

$$\pi^c_\sigma \otimes \pi^c_\tau \approx 2\left(\int_{\mathbf{R}^+} \pi_{it,\epsilon''}\, dt\right) \oplus \left(\bigoplus_{|k|\geqslant 2;k\,\text{even}} T_k\right).$$

For $\sigma + \tau < -1$,

$$\pi^c_\sigma \otimes \pi^c_\tau \approx 2\left(\int_{\mathbf{R}^+} \pi_{it,\epsilon''}\, dt\right) \oplus \left(\bigoplus_{|k|\geqslant 2;k\,\text{even}} T_k\right) \oplus \pi^c_{\sigma+\tau+1}.$$

(d) $$T_n \otimes \pi_{r,\epsilon} \approx \left(\int_{\mathbf{R}^+} \pi_{it,\epsilon\epsilon(n)}\, dt\right) \oplus \left(\bigoplus_{k\geqslant 2;k\equiv\epsilon\epsilon(n)} T_k\right),$$

$$T_n \otimes \pi^c_\sigma \approx \left(\int_{\mathbf{R}^+} \pi_{it,\epsilon(n)}\, dt\right) \oplus \left(\bigoplus_{k\geqslant 2;k\equiv\epsilon(n)} T_k\right).$$

(For tensor products with T_{-n}, replace T_k by T_{-k}.)

(e) $$T_m \otimes T_n \approx \bigoplus_{k=0}^{\infty} T_{m+n+2k}; \quad T_{-m} \otimes T_{-n} \approx \bigoplus_{k=0}^{\infty} T_{-m-n-2k}.$$

(f) $$T_m \otimes T_{-n} \approx \mathrm{Ind}_K^G\, \chi_{m-n} \approx \left(\int_{\mathbf{R}^+} \pi_{it,\epsilon(m-n)}\, dt\right) \oplus \left(\bigoplus T_k\right),$$

where the last summation is finite, involving those k with the same sign and parity as $m - n$, with $|m - n| \geqslant |k| \geqslant 2$. (The last isomorphism is trivial, by Frobenius reciprocity; see Anh [1].)

5. **Remarks.** To prove (b), (c) and (d), we consider an intertwining operator used by Martin [3] to prove (a), and note that for two representations of the nonunitary principal series this operator is still defined, though not a unitary isomorphism. Realizing complementary series (and discrete series) representations as unitarizations of (quotients of) nonunitary principal series representations, we apply Schur's Lemma to get a unitary isomorphism from tensor products of such representations to a known representation. In the second part of (c), the domain of the operator is not dense; the complementary subspace results in the extra summand. Part (e) is trivial, and (f) could be done by the method just outlined, but we have another method, based on an idea which was kindly suggested by Roger Howe. In a subsequent paper, we shall extend this method to other groups, and so-called holomorphic discrete series representations. In another paper we shall extend at least some of the above results to $\mathrm{SL}_2(K)$, where K is a locally compact totally disconnected nondiscrete field.

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DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NEW JERSEY 08540