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ON A NEW DEFINITION OF DERIVATIVE

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It is well known, since Weierstrass, that there are continuous functions that are not derivable at any point. The same is true for the various known generalizations of derivative, e.g. the unilateral, approximate and the symmetric derivatives. The present announcement deals with a new definition of derivative in terms of which every continuous function f is derivable at a *c-dense* set of points (viz. the set meets every interval in a set whose power is c), and the properties of f can be investigated in terms of the values of its new derivative wherever it exists.

Let $f: R \longrightarrow R$, where R denotes the set of real numbers. Let f be called *upper derivable* at a point $x \in R$ if $D^+f(x) \leqslant D_-f(x)$, and then an extended real number α is called an *upper derivative* of f at x if $D^+f(x) \leqslant \alpha \leqslant D_-f(x)$. Defining f to be *lower derivable* at x if $-f$ is upper derivable at x, it is clear that f is derivable at x if and only if it is upper and lower derivable there. These definitions can be easily extended to real-valued functions on any real topological vector space.

What is unusual about the upper and lower derivatives is that they are not unique like the derivative; consider e.g. $f(x) = |x|$ at $x = 0$. They are, however, unique at all but a countable set of points. Also, if f is *nonangular,* viz. $D_-f \leqslant D^+f$ and $D_+f \leqslant D^-f$ everywhere, then the upper or lower derivative of f is unique at every point where it exists. Such functions, in fact, form a residual set in the space $C[0, 1]$ with the uniform norm.

If f has a finite upper derivative at a point x, it is clearly upper semicontinuous at x. Every u.s.c. function f, on the other hand, has a finite upper derivative at a dense set of points, and this set becomes c-dense when f is nonangular. With the help of an analogue [1] of the Denjoy-Young-Saks theorem, we prove

THEOREM 1. *If f is continuous, then for almost every value of y in R, the level $f^{-1}(y)$ contains two dense sets of points where f has unique upper and*

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lower derivatives. Consequently, f has unique upper and lower derivatives at two c-dense sets of points on each of which f assumes almost all of its values at least once.

The Besicovitch function, which is nowhere right or left derivable, is upper and lower derivable at two metrically dense sets of points covering together almost all the points of the domain.

THEOREM 2. *An u.s.c. function f is nondecreasing whenever one of the following holds*: (a) f has *nowhere a finite upper derivative* < 0; (b) f *is nonangular and the set of points where f has a finite upper derivative* < 0 *has power* $< c$; (c) $\bar{D}f > 0$ *at a dense set of points and f has nowhere a zero upper derivative.*

THEOREM 3. *A function f is nondecreasing if, and only if,* $\bar{f}(x - 0) \leqslant f(x) \leqslant \bar{f}(x + 0)$ *at every point x, the upper derivative of f is* $\geqslant 0$ *at almost every point where it exists, and the points where f has a unique upper derivative* $-\infty$ *form a set whose power is* $< c$.

The last two theorems are clearly not possible for the ordinary, unilateral, approximate and symmetric derivatives. They yield sufficient conditions in terms of the upper derivative for a function to be constant or Lipschitz. There exist other similar results, leading in turn to results on the upper derivative of nowhere monotone and singular functions. Following is a strengthened form of the Goldowski-Tonelli theorem:

THEOREM 4. *Suppose f is of Baire class* 1 *and* $\underline{f}(x - 0) \leqslant f(x) \leqslant \bar{f}(x + 0)$ *at every point x. If f is upper derivable at all but a countable set of points and the ordinary derivative of f is* $\geqslant 0$ *at almost all of the points where it exists, then f is nondecreasing.*

When f is upper derivable at x, let $f'_u(x)$ denote the set of upper derivatives of f at x which is clearly a connected closed subset of the space $\bar{R}$ of extended real numbers.

THEOREM 5. *If two functions f and g have a pair of upper derivatives at x whose sum exists, then f + g is upper derivable at x and* $f'_u(x) + g'_u(x) \subset (f + g)'_u(x)$. *Identity holds when f or g has a finite derivative at x, or when the upper derivative of f + g at x is unique (which is true at all but a countable set of points).*

A similar result holds for the product of f and g, only the signs of $f(x)$ and $g(x)$ need to be taken into account. The chain rule also works out if one of the two functions has a nonzero finite derivative at the point in question. Relative to the exponential topology on the space $2^{\bar{R}}$ of closed subsets of $\bar{R}$, we have

THEOREM 6. *The upper derivative of every function f is of Baire class* 2 *relative to the set of points where it exists, and it is of Baire class* 1 *relative to the set of points where f is upper semicontinuous and uniquely upper derivable.*

Given a subset E of R, let a function $\phi: E \longrightarrow 2^{\bar{R}}$ be said to have the *Darboux property* if for every connected set C in R the set $\bigcup \{\phi(x): x \in E \cap C\}$ is connected in $\bar{R}$.

THEOREM 7. *Let f be an u.s.c. function with nowhere an ordinary discontinuity from any side.* (a) *If f is everywhere upper derivable, then its upper derivative has the Darboux and the mean-value properties, and it is an ordinary derivative at the points where it is unique and u.s.c. as a set-valued function.* (b) *If f is nonangular and upper derivable at all but a countable set of points, then the ordinary and upper derivatives of f both possess the Denjoy property relative to the sets of points where they exist.*

Thus a finite unique upper derivative is always of Baire class 1, it has the Darboux, mean-value and the Denjoy properties, and it is an ordinary derivative at the points where it is continuous.

Let f be called *semiderivable* at a point x if it is upper or lower derivable at x, and then the *semiderivative* $f_s'(x)$ of f at x is defined to be the upper or lower derivative of f at x. In case f is continuous, Theorems 4, 6 and 7 hold for semiderivative in place of the upper derivative. In fact, the semiderivative of a continuous function f possesses the Darboux and the mean-value properties without any hypothesis of semiderivability on f. Also, denoting by $f_s^{(n)}(x)$ the nth semiderivative of f at x, we have

THEOREM 8 (TAYLOR'S FORMULA). *If a continuous function f has n finite unique continuous semiderivatives on $[a, b]$, then it has an $(n+1)$th semiderivative α at some point in (a, b) such that*

$$f(b) = f(a) + (b-a)f_s'(a) + \cdots + \frac{(b-a)^n}{n!} f_s^{(n)}(a) + \frac{(b-a)^{n+1}}{(n+1)!}\alpha.$$

REFERENCES

1. K. M. Garg, *An analogue of Denjoy's theorem*, Ganita **12** (1961), 9–14. MR **26** #2556.

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