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EXISTENCE, UNIQUENESS, STABILITY FOR A SIMPLE FLUID WITH FADING MEMORY

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Let Ω be a bounded domain in R^3 with smooth boundary Γ. Let $v_i(\mathrm{x}, t)$ denote the velocity at a point $\mathrm{x} \in \Omega$ at time t of a simple fluid with fading memory when the strain relative to some fixed configuration is small (see [**1**, p. 90]). We assume the fluid is incompressible with density unity. Denote the stress tensor as S_{ij}, δ_{ij} the Kronecker delta, $\cdot$ to be $\partial/\partial t$. Consistent with [**1**] we choose as our constitutive equation

$$S_{ij} + p\delta_{ij} = 2\int_0^\infty m(s)\,[E_{ij}(t-s) - E_{ij}(t)]\,ds \tag{1}$$

where p is an indeterminate pressure, $m(s)$ a material function, E_{ij} the infinitesimal strain tensor. We are considering only a linear theory and must of consistency linearize the basic equation of motion,

$$\dot{v}_i + v_{i,j}v_j = S_{ij,j} \quad \text{in } \Omega, \tag{2}$$

to obtain as a linear model of a simple incompressible fluid with fading memory obeying (1), the equations:

$$\dot{v}_i = -p,_i + \int_0^\infty G(s)v_{i,jj}(t-s)\,ds \quad \text{in } \Omega, \tag{3a}$$

$$v_{j,j} = 0 \quad \text{in } \Omega \text{ (incompressibility)}, \tag{3b}$$

$$v_j = 0 \quad \text{on } \Gamma \text{ (viscous boundary condition)}, \tag{3c}$$

$$v_j(\mathrm{x}, \tau) = v_j^0(\mathrm{x}, \tau), \quad \mathrm{x} \in \Omega, -\infty < \tau \leqslant 0 \tag{3d}$$

$(v_j^0(\mathrm{x}, \tau)$ the initial velocity history).

Here $m(s) = dG(s)/ds$ where $G(s)$ is the shear relaxation modulus, $G(s) \longrightarrow 0$ as $s \longrightarrow \infty$.

Joseph [**2**] has noted that no mathematical theory presently exists for system (3a)–(3d). We have proven a positive existence, uniqueness, stability result in the case

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(i) $$G \in C^2[0, \infty) \cap L^1(0, \infty);$$

(ii) $$(-1)^k d^k G(s)/ds^k > 0, \ k = 0, 1;$$

(iii) $$d^2 G(s)/ds^2 \geqslant 0.$$

First we must give some notation.

Let $y_i(\mathrm{x}, s, t)$ denote the velocity history given by $y_i(\mathrm{x}, s, t) = v_i(\mathrm{x}, t - s)$ for $s \in R^+$. For pairs $(v_i(\mathrm{x}), y_i(\mathrm{x}, s))$ we define the inner product

$$\begin{aligned} &\langle\langle(\mathrm{v}, \mathrm{y}), (\mathrm{v}^*, \mathrm{y}^*)\rangle\rangle_H \\ &= \int_\Omega \left\{ \int_0^\infty G(s) y_i(\mathrm{x}, s)\, ds \right\}_{,j} \left\{ \int_0^\infty G(s) y_i^*(\mathrm{x}, s)\, ds \right\}_{,j} d\mathrm{x} \\ &\quad - \int_\Omega \int_0^\infty G'(s)(y_i(\mathrm{x}, s) - v_i(\mathrm{x}))(y_i^*(\mathrm{x}, s) - v_i^*(\mathrm{x}))\, ds\, d\mathrm{x}. \end{aligned}$$

Let H denote the Hilbert space obtained by completion with solenoidal vectors: $y_i \in C_0^\infty([0, \infty) \times \Omega)$, $v_i \in C_0^\infty(\Omega)$, $y_i(\mathrm{x}, 0) = v_i(\mathrm{x})$ in the H-inner product.

Define

$$A\begin{pmatrix} \mathrm{v} \\ \mathrm{y} \end{pmatrix} = \begin{pmatrix} \Delta_x \int_0^\infty G(s) \mathrm{y}(\mathrm{x}, s)\, ds \\ \\ - d\mathrm{y}(\mathrm{x}, s)/ds \end{pmatrix}$$

with

$$D(A) = \left\{ (\mathrm{v}, \mathrm{y}) \in H; A\begin{pmatrix} \mathrm{v} \\ \mathrm{y} \end{pmatrix} \in H \text{ and } \mathrm{y}(\mathrm{x}, 0) = \mathrm{v}(\mathrm{x}) \text{ for } \mathrm{x} \in \Omega \right\}.$$

Thus the differential equation

$$\frac{d}{dt}\begin{pmatrix} \mathrm{v}(t) \\ \mathrm{y}(t) \end{pmatrix} = A\begin{pmatrix} \mathrm{v}(t) \\ \mathrm{y}(t) \end{pmatrix}, \quad \begin{pmatrix} \mathrm{v}(0) \\ \mathrm{y}(0) \end{pmatrix} = \begin{pmatrix} \mathrm{v}^0 \\ \mathrm{y}^0 \end{pmatrix}$$

becomes an evolution equation representing (3a)–(3d). We can then state the following theorem.

THEOREM. *For $G(s)$ satisfying* (i), (ii), (iii), *A is the infinitesimal generator of a C_0 semigroup of contractions on H and the stability estimate $\|(\mathrm{v}(t), \mathrm{y}(t))\|_H \leqslant \|(\mathrm{v}^0, \mathrm{y}^0)\|_H$ is satisfied.*

The proof is based on an application of the Lumer-Phillips Theorem [3] for linear contraction semigroups and will appear elsewhere. It should be noted that existence and uniqueness of solutions to (3a)–(3d) may be obtained by other techniques if (3a)–(3d) is regarded as a Volterra integral equation. We mention in particular the work of Friedman and Shinbrot [4] and modification of the arguments of Barbu [5]. The advantage of our approach is that it yields simultaneously in addition to existence and uniqueness, a stability estimate in the "fading memory" norm H. We also note the recent paper of Nachlinger and Nunziato [6] which shows that sufficiently smooth solutions of (3a)–(3d) will have the property

$$\lim_{t\to\infty} \int_\Omega \frac{dv_i}{dt}(\mathrm{x}, t)\frac{dv_i}{dt}(\mathrm{x}, t)\,d\mathrm{x} = 0$$

under the more restrictive assumptions $G(s)$ positive, convex from below with $G'(s) + \xi G(s) \leqslant 0$, $G''(s) + \xi G'(s) \geqslant 0$ on $[0, \infty)$ for some number $\xi > 0$, and $G(s) \longrightarrow 0$ as $s \longrightarrow \infty$. Using arguments of elementary topological dynamics we are able to get a similar asymptotic stability result under the less restrictive assumptions $-\int_0^\infty G^1(s)s^2\,ds < \infty$, (i)–(ii).

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