BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 4, July 1976

RESULTANTS OF MATRIX POLYNOMIALS

BY I. C. GOHBERG AND L. E. LERER

Communicated by Chandler Davis, February 24, 1976

The $(n + m) \times (n + m)$ matrix

$$R(a, b) = \begin{bmatrix} a_0 & a_1 & \cdots & a_n & & & \\ & a_0 & a_1 & \cdots & a_n & & \\ & & \ddots & \ddots & & \ddots & \\ & & & a_0 & a_1 & \cdots & a_n \\ b_0 & b_1 & \cdots & b_m & & & \\ & b_0 & b_1 & \cdots & b_m & & \\ & & \ddots & \ddots & & \ddots & \\ & & & b_0 & b_1 & \cdots & b_m \end{bmatrix}$$

is called the *resultant matrix* of the two polynomials $a(\lambda) = a_0 + a_1\lambda + \cdots + a_n\lambda^n$ and $b(\lambda) = b_0 + b_1\lambda + \cdots + b_m\lambda^m$ $(a_j, b_j, \in \mathbf{C}^1, a_n \neq 0, b_m \neq 0)$. The determinant of this matrix is called the *resultant* of the polynomials $a(\lambda)$ and $b(\lambda)$. The following classical theorem on resultants is well known: *The number of common roots (counting multiplicities) of the polynomials $a(\lambda)$ and $b(\lambda)$ is equal to* dim Ker $R(a, b)$.

This statement does not admit a straightforward generalization to matrix polynomials [1], if the same definition of the resultant matrix $R(a, b)$ is used as in the one-dimensional case. For example the matrix

$$R\left(\begin{pmatrix} \lambda - 1 & 0 \\ 1 & \lambda - 1 \end{pmatrix}, \begin{pmatrix} \lambda & 1 \\ 0 & \lambda - 2 \end{pmatrix}\right)$$

is not invertible although the polynomial matrices do not have common eigenvalues, and the matrix

$$R\left(\begin{pmatrix} \lambda + 1 & 0 \\ 1 & \lambda \end{pmatrix}, \begin{pmatrix} \lambda & -1 \\ 0 & \lambda + 1 \end{pmatrix}\right)$$

AMS (MOS) subject classifications (1970). Primary 15A54; Secondary 15A24.
Key words and phrases. Linear algebra, resultant, matrix polynomial, resultant matrix, λ-matrix, common eigenvalue, multiplicity of a common eigenvalue.

is invertible although the eigenvalues of the polynomials coincide.

The main result of this article is concerned with a new definition of a resultant matrix $R^{\otimes}(a, b) \overset{\text{def}}{=} R(a \otimes I, I \otimes b)$, where $\otimes$ is the sign of the right-hand tensor product.

Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be all the common eigenvalues of $a(\lambda)$ and $b(\lambda)$. Let us denote by $k_{1p}(a) \geqslant k_{2p}(a) \geqslant \cdots \geqslant k_{j_p p}(a)$ the powers of the elementary divisors of $a(\lambda)$ for the eigenvalue λ_p. Let

$$\mu(a, b, \lambda_p) = \sum_{l=1}^{j_p(a)} \sum_{j=1}^{j_p(b)} \min\{k_{lp}(a), k_{jp}(b)\}$$

and

$$\mu(a, b) = (\mu(a, b, \lambda_1) + \mu(a, b, \lambda_2) + \cdots + \mu(a, b, \lambda_r)).$$

The main result is the following generalization of the classical resultant theorem.

THEOREM. *Let $a(\lambda)$ and $b(\lambda)$ be $d \times d$ matrix polynomials with invertible highest coefficients. Then* dim Ker $R^{\otimes}(a, b) = \mu(a, b)$. *Particularly $a(\lambda)$ and $b(\lambda)$ have a common eigenvalue if and only if* det $R^{\otimes}(a, b) = 0$.

We start sketching the proof by defining the *common multiplicity* of the eigenvalue λ_0 of polynomials $a(\lambda)$ and $b(\lambda)$. Let $\phi_0, \phi_1, \ldots, \phi_r$ be a chain of the eigenvector ϕ_0 and the associated vectors $\phi_1, \ldots, \phi_r$, which correspond to λ_0:

$$\sum_{j=1}^{k} \frac{1}{j!}\left(\frac{d^j}{d\lambda^j} a\right)(\lambda_0)\phi_{k-j} = \sum_{j=1}^{k} \frac{1}{j!}\left(\frac{d^j}{d\lambda^j} b\right)(\lambda_0)\phi_{k-j} = 0 \qquad (k = 0, 1, \ldots, r)$$

The number $r + 1$ is called the *length* of the chain. We denote the maximal length of such chain with the fixed vector ϕ_0 by rank (λ_0, ϕ_0). It is easy to find a basis $\phi_{10}, \phi_{20}, \ldots, \phi_{j_0,0}$ in the subspace $M = \operatorname{Ker} a(\lambda_0) \cap \operatorname{Ker} b(\lambda_0)$ such that $\operatorname{rank}(\lambda_0, \phi_{10}) = \max \operatorname{rank}(\lambda_0, \phi)$ $(\phi \in M)$ and $\operatorname{rank}(\lambda_0, \phi_{j0}) = \max \operatorname{rank}(\lambda_0, \phi)$ $(\phi \in M_j; j = 2, 3, \ldots, r)$ where M_j is the subspace with the basis $\phi_{j+1,0}, \phi_{j+2,0}, \ldots, \phi_{r,0}$. It is easy to see that for every vector $\phi \in M$, the integer $\operatorname{rank}(\lambda_0, \phi)$ is equal to one of the numbers $k_j = \operatorname{rank}(\lambda_0, \phi_{j0})$ $(j = 1, 2, \ldots, j_0)$. Therefore these numbers depend only on the polynomials $a(\lambda)$, $b(\lambda)$ and the eigenvalue λ_0. The integer $\nu(a, b, \lambda_0) = k_1 + k_2 + \cdots + k_{j_0}$ is called the *common multiplicity* of the eigenvalue λ_0 of the polynomials $a(\lambda)$ and $b(\lambda)$. If $M = \{0\}$, then we set $\nu(a, b, \lambda_0) = 0$.

The proof of the theorem involves two main steps. The first is to prove the equality $\mu(a, b, \lambda_0) = \nu(a \otimes I, I \otimes b, \lambda_0)$. The main theorem from **[1]** then implies that for large l, dim Ker $R_l(a \otimes I, I \otimes b) = \mu(a, b)$, where

$$R_l(a, b) = \left(\begin{array}{cccccc} a_0 & a_1 & \cdots & a_n & & \\ & a_0 & a_1 & \cdots & a_n & \\ & & \ddots & \ddots & & \ddots \\ & & & a_0 & a_1 \cdots & a_n \\ b_0 & b_1 & \cdots & b_m & & \\ & b_0 & b_1 & \cdots & b_m & \\ & & \ddots & \ddots & & \ddots \\ & & & b_0 & b_1 \cdots & b_m \end{array} \right) \begin{array}{l} \left.\vphantom{\begin{array}{c} a \\ a \\ \ddots \\ a \end{array}}\right\} n + l \\ \left.\vphantom{\begin{array}{c} b \\ b \\ \ddots \\ b \end{array}}\right\} m + l \end{array}$$

The second step consists of proving that the dim Ker $R_l(a \otimes I, I \otimes b)$ does not depend on l and therefore

$$\dim \operatorname{Ker} R_l(a \otimes I, I \otimes b) = \dim \operatorname{Ker} R(a \otimes I, I \otimes b).$$

Let us mention that the main theorem is connected with the theory of the equation $a(\lambda)\, x(\lambda) + y(\lambda)\, b(\lambda) = f(\lambda)$, where $f(\lambda)$ is the given and $x(\lambda)$, $y(\lambda)$ are the unknown matrix polynomials.

All the detailed proofs will appear elsewhere together with some analogous results for the continuous case.

REFERENCES

1. I. C. Gohberg and G. Heining, *Resultant matrix and its generalization.* I: *Resultant operator of matrix polynomials,* Acta Sci. Math. (Szeged) **37** (1975), Fasc. 1–2, pp. 41–61. (Russian)

DEPARTMENT OF MATHEMATICAL SCIENCES, TEL-AVIV UNIVERSITY, RAMAT-AVIV, ISRAEL

DEPARTMENT OF MATHEMATICS, WEIZMANN INSTITUTE OF SCIENCE, REHOVOT, ISRAEL

DEPARTMENT OF MATHEMATICS, TECHNION, HAIFA, ISRAEL