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FINITE HJELMSLEV PLANES WITH NEW INTEGER INVARIANTS

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Projective Hjelmslev planes (PH-planes) are a generalization of projective planes in which each point-pair is joined by at least one line and, dually, each line-pair has a nontrivial intersection. Multiply joined points (and multiply intersecting lines) are called neighbor points (and neighbor lines). By hypothesis, the neighbor relations of a PH-plane A are equivalence relations which induce a canonical epimorphism from A to a projective plane $\bar{A}$. If A is finite, there exists [4] an integer t such that the inverse image of every point and every line of $\bar{A}$ contains precisely t^2 elements. If the order of $\bar{A}$ is r, we say that A is a (t, r) PH-plane. We are concerned with the problem of determining the spectrum S of all admissible pairs (t, r). Since the finite projective planes are simply the $(1, r)$ PH-planes, our concern is with a generalization of the classical existence question for projective planes.

Prior to this announcement, the only pairs (t, r) known to belong to S satisfy the requirements:

(1) t is a power of r,

(2) r is a prime power.

Conversely, all such pairs do belong to S, and all arise as the invariants of the Desarguesian-Pappian PH-planes investigated by Klingenberg [5]. A deep theorem of Artmann [1] allows one to assert that (t, r) is in S if (1) holds and if r is the order of a projective plane. Whether this is any improvement over the previous result is, however, still uncertain.

Nonexistence results to date are also few in number. The celebrated Bruck-Ryser Theorem gives infinitely many values of r for which $(1, r) \notin S$. Clearly $(1, r) \notin S$ implies $(t, r) \notin S$ for any t. Kleinfield [4] has observed that $(t, r) \in S$ with $t \neq 1$ implies $t \geqslant r$. Most recently, Drake [2] has proved that $(t, r) \in S$ with $1 \neq t \neq r$ implies that $t = 4$ or 8 or that $r \leqslant t + 1 - \sqrt{(2t + 3)}$.

The current note is written to announce the following two existence results:

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THEOREM 1. *Let t, r, q, b be positive integers such that $(t, r) \in S$, q is a prime power and $q + 1 = t(r + 1)$. Then $(t \cdot q^b, r) \in S$.*

THEOREM 2. *Let t, q, b be positive integers such that $(t, t) \in S$, q is a prime power and $2(t + 1) \leqslant q + 1 \leqslant t(t + 1)$. Then $(t \cdot q^b, t) \in S$.*

Details will be given elsewhere of a construction which simultaneously yields both theorems and a little more. Theorem 1, for example, allows one to conclude that $(8 \cdot 23^b, 2) \in S$ for arbitrary b. The actual Lenz-Drake construction applied to the "extremal" (8, 2) PH-planes of Shult and Drake [3] yields the additional information that $(8 \cdot 19^b, 2), (8 \cdot 17^b, 2) \in S$.

We remark that Theorem 1 may be applied either recursively or in tandem with Theorem 2. For example, since $(2, 2) \in S$, Theorem 1 (or 2) yields $(2 \cdot 5^b, 2) \in S$. A second application of Theorem 1 then yields $(2 \cdot 5 \cdot 29^d, 2)$, $(2 \cdot 25 \cdot 149^d, 2) \in S$.

The construction is largely elementary. We mention several of the basic ideas, presenting them in generality sufficient for the proof of Theorem 1. If $M = [m_{ij}]$ is an incidence matrix for a (t, r) PH-plane A, then every row and every column of M contains precisely $t(r + 1)$ one's. Thus König's Lemma implies that M is a sum of permutation matrices; consequently, it is possible to obtain a matrix $N = [n_{ij}]$ of the same size as M such that $n_{ij} = 0$ precisely when $m_{ij} = 0$ and so that every integer from 1 to $t(r + 1)$ appears in each row and each column of N. Next one seeks a suitable set of $t(r + 1)$ square matrices B_1, $B_2, \ldots$ of order s^2 where $s = q^b$. One then obtains a matrix G from N by substituting B_i for i when $i \geqslant 1$ and replacing each 0 by the square zero matrix of order s^2. For G to represent the desired $(t \cdot s, r)$ PH-plane, it suffices to demand that the matrices B_i satisfy:

$$B_i \cdot (B_j)^T = (B_j)^T \cdot B_i = J \quad \text{when } i \neq j \tag{3}$$

and

$$\sum B_i \cdot (B_i)^T, \quad \sum (B_i)^T \cdot B_i \geqslant 2J; \tag{4}$$

here J denotes the matrix of all one's, and one writes $[x_{ij}] \geqslant [y_{ij}]$ to mean that $x_{ij} \geqslant y_{ij}$ for all i, j.

The B_i can be obtained from an (s, q) PH-plane A' of the type investigated by Klingenberg and mentioned above. One obtains an incidence matrix $D = [D_{ij}]$ for A' so written that every D_{ij} is square of order s^2 and successive sets of s^2 columns (rows) represent neighbor classes of lines (points). Let $E_1, E_2, \ldots$ be the nonzero matrices among the D_{1x}; $F_1, F_2, \ldots$ be the nonzero matrices a among the D_{x1}. Then there exist permutation matrices P_i, Q_i such that $E_iP_i = Q_iF_i \equiv B_i$ for all i, and these B_i satisfy conditions (3) and (4).

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