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TOPOLOGICAL STABILITY IN THE NICE DIMENSIONS ($n \leqslant p$)

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0. **Introduction.** By [2], [3], and [5], in the nice dimensions the C^0 and C^∞-stable mappings give two classifications for open dense subsets of the space of smooth mappings. Two natural questions about the relations between these classifications are: (1) Can distinct C^∞-stable mappings become equivalent under the weaker notion of topological equivalence? (2) Which topologically stable mappings are C^∞-stable? When n (= dim domain) $\leqslant p$ (= dim range), we can answer: (1) not only does $Q(f)$ determine the C^∞-type of a C^∞-stable map germ f, but also $Q(f) \otimes_{\mathbf{R}} \mathbf{C}$ is an invariant of the topological type of f. (2) For topologically stable mappings $f: N \longrightarrow P$ with N compact, f is C^∞-stable $\Leftrightarrow$ any suspension of f is also topologically stable.

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1. **Topological invariance of $Q(f) \otimes_{\mathbf{R}} \mathbf{C}$.** A C^∞-stable map-germ $f: \mathbf{R}^n \longrightarrow \mathbf{R}^p$ is of *simple type* if its orbit under the action of $\mathcal{K}^{p+1}$ has a neighborhood containing only finitely many $\mathcal{K}^{p+1}$-orbits. This includes the types in the nice dimensions.

THEOREM 1. *For C^∞-stable map germs f of simple type, $Q(f) \otimes_{\mathbf{R}} \mathbf{C}$ is a topological invariant.*

COROLLARY 1. *For C^∞-stable germs f: $\mathbf{R}^n \longrightarrow \mathbf{R}^p$ of simple type the following are topological invariants*: $\delta(f)$, $\gamma(f)$, $\mu_{n-p}(f)$, $\operatorname{codim}(\mathcal{K} \cdot f)$, *Thom-Boardman symbols* (*using the notation of* [2, VI]).

Theorem 1 is probably true even for the C^∞-stable germs of discrete algebra type. Although the theorem does not say that $Q(f)$ is a topological invariant in the nice dimensions this is very nearly true as in most cases there is only one real $Q(f)$ for each $Q(f) \otimes_{\mathbf{R}} \mathbf{C}$ type.

2. **S-stable maps and their properties.** Next, we want to describe the topologically stable mappings in the nice dimensions that are C^∞-stable. We know

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that C^∞-stable mappings are topologically stable but they also remain so under suspension, i.e. if $f: N \longrightarrow P$ is C^∞-stable, then so is $f \times 1_M: N \times M \longrightarrow P \times M$.

DEFINITION. *A C^∞-mapping $f: N \longrightarrow P$ is S-stable if all $f \times 1^k: N \times (S^1)^k \longrightarrow P \times (S^1)^k$ are topologically stable. Throughout we assume that N is compact.* (This is necessary, see [4].)

PROPOSITION 2. *S-stable mappings are transverse to the simple K^∞-orbits.*

For the next proposition we have to define the immersion condition [4].

DEFINITION. *Let $\Gamma \subset J^l(m, q)$, $q - m = k$, be a singularity submanifold invariant under K^l; then we say Γ satisfies the immersion condition if for any C^∞-stable mapping $f: N \longrightarrow P$ with $p - n = k$, $f | \Gamma(f)$: $\Gamma(f) \longrightarrow P$ is a C^∞-immersion*

PROPOSITION 3. *Suppose $\Gamma \subset J^l(m, q)$ is a singularity submanifold invariant under K^l such that* (1) *there is an open dense $\Gamma_0 \subset \Gamma$ which satisfies the immersion condition for $q - m = k$.* (2) *Γ-type is a topological invariant for C^∞-stable map germs, when $q - m = k$. Then if $f: N \longrightarrow P$ is an S-stable mapping with $p - n = k$ and $n <$ codim Γ then $\Gamma(f) = \varnothing$.*

Then, after an analysis of the sets $\Pi = \Pi(n, p)$ [2, VI], we have

PROPOSITION 4. *If $f: N \longrightarrow P$ is an S-stable mapping with (n, p) in the nice dimensions, then $\Pi(f) = \varnothing$, and f is multitransverse to the simple orbits.*

Finally, putting these three propositions together, we can conclude

THEOREM 5. *In the nice dimensions ($n \leqslant p$), (N compact) S-stability is equivalent to C^∞-stability.*

3. **Outline of proof.** Theorem 1 follows by the methods of [1].

For the theorems on the properties of S-stable maps we use a modification of the method used by May [4]. May perturbed a nontransverse map to make it nontransverse in a specific way. We modify this idea because our topological classification (Theorem 1 and [1]) is only valid for C^∞-stable map germs.

PROPOSITION. *Let $f: N \longrightarrow P$ be S-stable with $n =$ dim N sufficiently big, and $\Gamma \subset J^l(N, P)$ a singularity submanifold satisfying the conditions of Propositions 2 or 3. If $j^l(f)$ is not transverse to Γ at a point $x \in N$, then there are f_1, f_2, topologically stable and equivalent to f, and neighborhoods $U_1 \subset U_2$ of x such that:* (1) *$f_1 = f_2$ outside of U_1;* (2) *f_1, f_2 consist of C^∞-stable map germs at points $x' \in U_2$ except at one point of U_2;* (3) *$\Gamma^0(f_1) \cap U_1$ is homeomorphic to $C - \{0\}$ where C is a cone $\Sigma_{i=1}^r \lambda_i x_i^2 = 0$ and $\Gamma^0(f_2) \cap U_1$ is homeomorphic to the generalized hyperboloid $\Sigma_{i=1}^r \lambda_i x_i^2 = \epsilon$.*

$\Gamma^0(f) = \{x \in N \mid f_{(x)}$ is topologically equivalent to a C^∞-stable map-germ of Γ-type$\}$. Then, we use a Mayer-Vietoris argument to compare $H_*(\Gamma^0(f_1))$ and $H_*(\Gamma^0(f_2))$. Then, by the proposition, we can show these are different once we know they are finitely generated groups.

For this we use the following proposition: if g is a C^∞-map germ, define $\Sigma_g^0(f) = \{x \in N \mid f_{(x)}$ is topologically equivalent to $g\}$.

PROPOSITION. *If $f: N \longrightarrow P$ is a topologically stable mapping (N compact), then $\Sigma_g^0(f)$ is a finite (disjoint) union of manifolds with finitely generated homology.*

Then, we can use Proposition 3 to prove that if $f: N \longrightarrow P$ is S-stable, then $\Pi(f) = \varnothing$. This follows because any germ in $\Pi(n, p)$ can be deformed to one of the following types: (1) Σ_i, $i > 5$, (2) $\Sigma_{5,(1)}$, (3) $\Sigma_{4,(2)}$, (4) $\Sigma_{3,(3)}$, (5) (x^2, y^4) in $\Sigma_{2,1}$ or (6) $\Sigma_{2,2,(1)}$.

However, these types are topological invariants and Proposition 3 implies that they do not occur in the nice dimensions. Thus, neither does any $\Pi(n, p)$ type. Finally, for multitransversality, we apply the same method but to the images of the singularity sets in P.

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