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THE HOLOMORPHIC LEFSCHETZ FORMULA

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Let X be a compact complex manifold and let $f: X \longrightarrow X$ be a holomorphic map. One can assign to each component Y of the fixed point set F of f a complex number $\nu_Y(f)$ so that

$$L(f, \mathcal{O}) = \sum_Y \nu_Y(f).$$

$L(f, \mathcal{O})$ denotes the Lefschetz number on $H^*(X, \mathcal{O})$. In this note we outline a computation of $\nu_Y(f)$ in the case that Y is a *nondegenerate* component of F, i.e., Y is a *submanifold* of X, and if df^N denotes the map induced by df on the normal bundle of Y, then $\det(1 - df^N) \neq 0$. Our result is that $\nu_Y(f)$ is given by the same formula proved by Atiyah and Singer [3] in the case that f is an isometry. If dim $Y = 0$ the formula was known without this restriction on f by Atiyah and Bott [2]. Patodi [7] was able to remove the restriction on f under other assumptions, which are vacuous if dim $Y = 1$.

Our methods are purely algebraic, and go through in algebraic geometry of characteristic zero. In particular, for $f =$ identity, we obtain a simpler justification of the local formula used in [8] to prove the Riemann-Roch theorem that is also valid in the algebraic category.

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1. **Statement of the formula.** Let N denote the normal bundle of Y and let $\lambda_1, \ldots, \lambda_m$ be the eigenvalues of df^N. N splits as direct sum of bundles N_i of dimension d_i on which $df^N - \lambda_i 1$ is nilpotent. Then the component of degree zero of the characteristic class $\Sigma_{p=0}^{d_i} (-1)^p \lambda_i^p \operatorname{ch}(\Lambda^p N_i^*)$ is $(1 - \lambda_i)^{d_i} \neq 0$, hence this class is invertible in the cohomology ring of Y. The formula for $\nu_Y(f)$ is

$$\nu_Y(f) = \int_Y \mathcal{T}(Y) \left\{ \prod_{i=1}^{m} \sum_{p=0}^{d_i} (-1)^p \lambda_i^p \operatorname{ch}(\Lambda^p N_i^*) \right\}^{-1} \tag{1.1}$$

$\mathcal{T}(Y)$ is the total Todd class of Y and the integral sign denotes evaluation on the fundamental cycle. We always think of characteristic classes as taking values in

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the ring $H^*(Y, \Omega^*)$ via Atiyah's definition [1]. If f is covered by a map φ of a bundle E over X, we also have a local index $\nu_Y(f, \varphi, E)$ for the action on $H^*(X, O(E))$ which is given by decomposing $E|Y$ into bundles E_i on which $\varphi - \mu_i 1$ is nilpotent, and inserting the factor $\Sigma \mu_i \mathrm{ch}(E_i)$ under the integral sign in (1.1).

2. **Outline of the proof.** Let $\Omega^{0,n}$ be the sheaf $\pi_1^* O \otimes \pi_2^* \Omega_X^n$ over $X \times X$ ($n = \dim X$). The machinery of local cohomology [6, §§1, 2] and Grothendieck's duality theory [5] give us natural maps

$$\text{(2.1)} \qquad \mathrm{Ext}^n(O_\Delta, \Omega^{0,n}) \longrightarrow H^n_\Delta(X \times X, \Omega^{0,n}) \longrightarrow H^n(X \times X, \Omega^{0,n})$$

and in each one of these groups a class dual to Δ, which we denote by Δ', Δ'', Δ''' respectively. If Γ is the graph of f: $\Gamma x = (fx, x)$, then $\Gamma^*\Delta''' \in H^n(X, \Omega^n)$ gives the Lefschetz number and $\Gamma^*\Delta'' \in H^n_F(X, \Omega^n) \approx \bigoplus_Y H^n_Y(X, \Omega^n)$ gives the local indices $\nu_Y(f)$.

In general, $\Gamma^*\Delta'$ does not make sense. But if Y is *nondegenerate*, U, V suitable neighborhoods of Y, then we can complete the diagram

$$\text{(2.2)} \qquad \begin{array}{ccc} \mathrm{Ext}^n(O_Y, \Omega^n) & \stackrel{\Gamma^*}{\longleftarrow\cdots\cdots} & \mathrm{Ext}^n(O_\Delta, \Omega^{0,n}) \\ \downarrow & & \downarrow \\ H^n_Y(V, \Omega^n) & \stackrel{\Gamma^*}{\longleftarrow} & H^n_\Delta(U \times V, \Omega^{0,n}). \end{array}$$

Geometrically this means that, under the nondegeneracy condition, Γ^* preserves the natural filtrations of H_Δ, H_Y given by the order of poles. The canonical isomorphism

$$\mathrm{Res}\colon \mathrm{Ext}^n(O_Y, \Omega^n_X) \xrightarrow{\approx} H^r(Y, \Omega^r_Y)$$

($r = \dim Y$) gives the formula

$$\text{(2.3)} \qquad \nu_Y(f) = \int_Y \mathrm{Res}\, \Gamma^*\Delta'.$$

Our proof then proceeds as follows:

(i) We give a concrete description of $\mathrm{Ext}(O_\Delta, \Omega^{0,n})$, $\mathrm{Ext}(O_Y, \Omega^n)$ as the cohomology of a "twisted Čech complex." This complex is actually hidden in [8, §§8–10], and the starting point is the similarity of Lemma (10.7) there with the twisting cochains of [4]. In this complex there is a canonical cocycle representing Δ', namely the sequence $\{\tau^p\}$ of [8, Corollary (10.22)].

(ii) We then construct a chain map of these complexes that induces the map Γ^* of (2.2), thus a formula for $\nu_Y(f)$ via (2.3).

(iii) Since all our constructions are by explicit local formulas, extending the arguments of [8, §§5, 6] we obtain

$$\text{(2.4)} \qquad \nu_Y(f, \varphi, E) = \int_Y \omega(f)\mathrm{ch}(\varphi, E)$$

where $\omega(f)$ depends only on the action of df on N and $\mathrm{ch}(\varphi, E)$ is the "Chern character" of φ.

(iv) The integrands in (2.4) and (1.1) cannot agree in general, because (1.1) depends only on the semisimple parts of df, φ, while (2.4) depends also on the nilpotent parts. To show that the integrated formulas agree we reduce the problem to the fact that they agree when $f = \mathrm{id}$, φ arbitrary (Riemann-Roch theorem) by studying the action of df on suitable sheaves over $P(N \oplus 1)$.

3. **Other remarks.** The twisted Čech complex of §2 can be used to construct dual classes in more familiar Čech complexes; it is in fact the universal complex for constructions of this type. For instance, one can interpret local cohomology as relative Čech cohomology, and the first map of (2.1) can be realized by an explicit chain map. Its construction is by necessity quite complicated and equivalent to the inductive arguments of [8, §10]. But this point of view makes the computations there much more plausible.

Using this map we can write down explicit formulas for the dual class of a submanifold of any codimension, generalizing the familiar formulas for codimension one.

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