

A SOLUTION TO THE BLUMBERG PROBLEM

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In 1922 Henry Blumberg proved

PROPOSITION 1 [1]. *If X is a separable complete metric space and $f: X \rightarrow \mathbf{R}$ is any real-valued function on X , then there exists a set D which is dense in X , such that the restriction of f to D is continuous.*

DEFINITION. Call a space X *Blumberg* iff for any real-valued function f on X , there exists a dense subspace $D \subseteq X$ such that $f \upharpoonright D$ (the restriction of f to D) is continuous.

Recall that a space X is a *Baire* space iff no open subset of X is the union of countably many nowhere dense subsets. In 1960 J. Bradford and C. Goffman improved Blumberg's results, establishing

PROPOSITION 2 [2]. *A metric space X is Blumberg iff it is a Baire space.*

The question arose: "Which Baire spaces are Blumberg?" and, in particular, the Blumberg problem: "Must every compact Hausdorff space be Blumberg?" Partial answers were given by R. Levy [5], [6] and H. E. White [8]. In fact, there *is* a non-Blumberg compact Hausdorff space. An unusual feature of our example is that it is the disjoint union of two spaces, one or the other of which fails to be Blumberg, depending on whether or not the continuum hypothesis holds.

Let B be the Boolean algebra of Lebesgue measurable subsets of $[0, 1]$, and let I be the ideal of null sets of B . The reduced measure algebra is the complete Boolean algebra B/I . Let $\text{St}(B/I)$ be the Stone space of the reduced measure algebra. As is shown in [3], $\text{St}(B/I)$ is a compact Hausdorff, extremally disconnected space in which first category sets are nowhere dense. Also, it has weight 2^{\aleph_0} and satisfies the countable chain condition. It can be shown that $\text{St}(B/I)$ is the union of 2^{\aleph_0} nowhere dense sets. From these facts, we obtain the following theorem.

THEOREM 1. *Assume $2^{\aleph_0} = \aleph_1$. $\text{St}(B/I)$ is not Blumberg.*

DEFINITION. A collection \mathcal{C} of open subsets of a space X is *oblivious* iff there exists an open subset V of X such that $(\forall x \in V)(\forall \mathcal{C}' \subseteq \mathcal{C})(x \in \bigcap \mathcal{C}'$ implies $(\exists$ open $W \subseteq V)(\emptyset \neq W \subseteq \bigcap \mathcal{C}')$.

THEOREM 2. *If X is a Baire LOTS (linearly ordered topological space),*

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then X is not Blumberg iff there exists an open $U \subseteq X$ such that

- (i) U is the union of $\leq 2^{\aleph_0}$ nowhere dense sets, and
- (ii) every countable collection of open subsets of U is oblivious.

The next theorem is obtained by modifying two known results: the classical construction of κ -Aronszajn trees (see [4]), and the "Souslin tree gives Souslin line" construction in [7].

THEOREM 3. *There exists a compact LOTS L with the following properties:*

- (a) *it is the union of \aleph_2 nowhere dense subsets,*
- (b) *any countable collection of open subsets of L is oblivious.*

Theorem 2, Theorem 3, and $2^{\aleph_0} \geq \aleph_2$ together imply that L is a non-Blumberg compact LOTS.

THEOREM 4. *There exists a compact Hausdorff space which is not Blumberg.*

PROOF. Let X be the disjoint union of $\text{St}(B/I)$ and L . If X is Blumberg, both $\text{St}(B/I)$ and L must be Blumberg. However, regardless of the value of 2^{\aleph_0} , $\text{St}(B/I)$ and L cannot both be Blumberg.

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