BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 4, July 1975

DEFORMING P.L. HOMEOMORPHISMS ON A CONVEX 2-DISK

BY CHUNG-WU HO

Communicated by Philip T. Church, March 3, 1975

1. **The main result.** Let D be a convex disk in R^2 whose boundary is a polygon. By a *triangulation* of D, we mean a (rectilinear) simplicial complex which has D as its underlying space. We shall call a homeomorphism f of D onto D a p.l. *homeomorphism* if there exists a triangulation K of D such that the restriction of f to each simplex σ of K is a linear map of σ into R^2. We shall consider only those p.l. homeomorphisms of D which are pointwise fixed on the boundary of D. In this note, we announce the following result.

THEOREM A. *For each p.l. homeomorphism f of D, there exists a triangulation K of D such that f may be realized by successively moving the vertices of K in a finite number of steps (with the motion being extended linearly to each simplex of K) such that in the process of moving, none of the simplices is allowed to collapse.*

The general problem of deforming a prescribed map of a space into the identity map, or vice versa, in a specific manner has a long history. For the special case of deforming a particular homeomorphism of an n-cell into the identity map through a special class of homeomorphisms, H. Tietze proved as early as 1914 that any homeomorphism of a 2-disk, which is pointwise fixed on the boundary of the disk, can be deformed into the identity map through a family of such homeomorphisms [5]. This result was extended in 1923 for an n-dimensional cell by J. W. Alexander [1]. The technique used by Alexander can in fact be used to show that each p.l. homeomorphism on a polyhedral n-cell, which is pointwise fixed on the boundary of the cell, can be deformed into the identity map through a family of such p.l. homeomorphisms. However, each of the p.l. homeomorphisms of the family requires a different triangulation of the domain space. It is therefore natural to ask whether this

AMS (MOS) subject classifications (1970). Primary 57C99; Secondary 57C25, 57C35.
Key words and phrases. Simplicial complex, triangulation, proper triangulation, p.l. homeomorphism, deformation of maps, simplicial subdivision.

deformation can be carried out in such a way that all the p.l. homeomorphisms required in the deformation process are linear with respect to a *fixed* triangulation of the n-cell. Our Theorem A clearly answers this question in the affirmative for a convex 2-dimensional polyhedral disk. Our proof of Theorem A, as outlined in the next two sections, is based on the assumptions that the disk is convex and 2-dimensional. We do not know whether the theorem is still true for a higher dimensional disk or for a 2-dimensional disk which is not convex.

2. **Preliminaries.** For each triangulation K of D, we shall let $L(K)$ be the space of all p.l. homeomorphisms of D which are linear with respect to K. The space $L(K)$ is equipped with the compact open topology. Observe that each element $f \in L(K)$ is completely determined by the image under f of the vertices of K which are contained in the interior of D. Thus, if an ordering, say from 1 to n, is assigned to these interior vertices of K, each element f of $L(K)$ may be identified as a point in the space R^{2n}. In fact, one may establish without too much effort

PROPOSITION 1. *For each triangulation K of D, the space $L(K)$ may be identified as an open subset of R^{2n} where n is the number of vertices of K contained in the interior of D.*

Under this identification of $L(K)$ as an open subset of R^{2n}, we observe that each element $f \in L(K)$ has a neighbourhood N in $L(K)$ such that each $g \in N$ can be obtained from f by successively moving the images $f(v)$ of the vertices v of K. To see this, one needs only to construct a "cubic box" centered at f in R^{2n} which is contained in $L(K)$. Then one may deform f to any other element g of the box by moving successively the component of f in each copy of R^2 to the corresponding component of g. From this observation and an elementary compactness argument, one establishes immediately

PROPOSITION 2. *Let K be a triangulation of D and let f, g be two elements of $L(K)$. The element f may be deformed to g by moving successively the images $f(v)$ of the vertices of K if and only if f may be connected to g by a path in the space $L(K)$.*

With this proposition, one may rewrite our Theorem A in the following equivalent form.

THEOREM B. *For each p.l. homeomorphism f of D, there exists a triangulation K of D such that $f \in L(K)$ and f may be connected to the identity map of D by a path in $L(K)$.*

3. **Sketch of the proof of Theorem B.** Let K be a triangulation of D. A vertex of K is called a *boundary vertex* if it is contained in $\mathrm{Bd}(D)$. The triangulation K will be called a *proper* triangulation if no three boundary vertices of K are on a straight line. Intuitively, a proper triangulation of D has no vertices on the sides of D except at the "corners" of D. We first establish a special case of Theorem B, the case when the p.l. homeomorphism f belongs to $L(K)$ for a proper triangulation K of D (cf. [2]).

PROPOSITION 3. *The space $L(K)$ is pathwise connected for a proper triangulation K of D.*

We may then use an argument similar to that described in [3] to show that for each p.l. homeomorphism f of D, there is a proper triangulation K of D such that f may be connected to $L(K)$ by a path in some larger space $L(K')$. This implies Theorem B. The details of all the proofs will appear in [4].

REFERENCES

1. J. W. Alexander, *On the deformation of an n-cell*, Proc. Nat. Acad. Sci. U.S.A. 9 (1923), 406–407.

2. C. W. Ho, *On certain homotopy properties of some spaces of linear and piecewise linear homeomorphisms.* I, Trans. Amer. Math. Soc. 181 (1973), 213–233. MR 48 #1252.

3. ———, *On certain homotopy properties of some spaces of linear and piecewise linear homeomorphisms.* II, Trans. Amer. Math. Soc. 181 (1973), 235–243. MR 48 #1252.

4. ———, *Deforming p.l. homeomorphisms on a convex polygonal 2-disk*, Pacific J. Math. 57 (1975).

5. H. Tietze, *Über stetige Abbildungen einer Quadrätflache auf sich selbst*, Rend. Circ. Mat. Palermo 38 (1914), 247–304.

DEPARTMENT OF MATHEMATICAL STUDIES, SOUTHERN ILLINOIS UNIVERSITY, EDWARDSVILLE, ILLINOIS 62025