tial fraction expansion for $1/p(z)$. This idea can be made useful in various ways, particularly by Rutishauser's *qd* algorithm, which is developed in full detail and applied to entire functions as well as to polynomials. Anyone seeking information on how to calculate zeros will be well advised to consult these two chapters.

Some years ago there was a college president who said that he could accept functional architecture as long as it was "functional for use." This superficially fatuous remark makes, on reflection and considering the vogue use of the adjective, a good deal of sense. In the same vein, Henrici's book is about applied complex analysis for use. By using topics from it, any course that is oriented toward applications can be made more realistic and more useful; an abstract course that nevertheless acknowledges the mundane utility of the subject by including some applications can do so more intelligently. There is surely satisfaction in knowing whether one has to do with a pure existence result or with one that makes it possible to calculate something reasonably accurately in a reasonable amount of time. The algorithmic approach has also led to the formulation of a number of results in simpler or more elegant forms than are usually given. I think that Henrici has shown that his approach has a good deal to contribute to our understanding of complex analysis.

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Buildings of spherical type and finite BN-pairs, by Jacques Tits, Lecture Notes in Mathematics, no. 386, Springer-Verlag, Berlin, Heidelberg, New York, 1974, 299+x pp., \$9.90

The relationships between certain algebraic, analytic and geometric structures and root systems in Euclidean spaces have been a source of methods and ideas that have had a profound impact on various parts of mathematics. Some particularly fruitful instances of this interaction are E. Cartan's classification of semisimple Lie algebras over the complex field [**4**], H. Weyl's papers on the representations of semisimple Lie algebras and Lie groups [**9**], E. Cartan's work on symmetric spaces (see [**7**]), H. S. M. Coxeter's enumeration of finite groups generated by reflections (see [**2**]), C. Chevalley's Tôhoku paper [**5**] and his classification of semisimple algebraic groups [**6**], and Harish-Chandra's contributions to Fourier analysis on semisimple Lie groups (see [**8**]). There are many others. It is not my purpose to give here a historical survey of this subject, but simply to remind the reader of some high points in the development of the theme, and to state that the volume which is the subject of this review belongs in this distinguished company.

The present volume is primarily a research article, devoted to the proofs of two main results. The mathematical community has been aware of the existence of these theorems for some time, and the author has announced parts of the work on several previous occasions. These notes contain the first detailed proofs of the main results. Their importance is difficult to overemphasize. They provide a common combinatorial foundation for several topics mentioned in the first paragraph, and are remarkable both for the originality and beauty of their conceptual framework, and for the depth and power required for the complete analysis of the possibilities.

The buildings of spherical type, whose classification is a main purpose of the notes, are certain complexes, or partially ordered sets. The buildings satisfy axioms to which the author was led in his search for a systematic procedure for the geometrical interpretation of semisimple Lie groups and, in particular, the exceptional groups. A model for this procedure is afforded by the way complex projective space is recovered from the group $\mathrm{SL}_n(C)$, for $n \geqq 3$. The first step is to identify a family of subgroups of $\mathrm{SL}_n(C)$, called parabolic subgroups. These may be described in various ways, as containing the stabilizers of flags in the vector space, or as closed subgroups P for which the quotient algebraic variety $\mathrm{SL}_n(C)/P$ is complete, or as subgroups containing maximal connected solvable subgroups. The subspaces in the geometry associated with $\mathrm{SL}_n(C)$ are the maximal parabolic subgroups. Two subspaces are incident in the projective space if the intersection of the corresponding parabolic subgroups is parabolic. This method of constructing a geometry from a group can be applied to any complex semisimple Lie group, in such a way that the basic properties of the associated geometry are directly related to the root system and the Weyl group of the Lie group in question. The same method can be applied to semisimple algebraic groups, and to the Chevalley groups [**5**], in which the underlying field is quite arbitrary.

The theory of buildings has found applications which go far beyond the initial purpose of interpreting geometrically the exceptional groups. Buildings of spherical and affine type have been useful in the study of

p-adic simple groups, the cohomology of arithmetic groups, the representation theory and harmonic analysis on finite and p-adic simple groups, and some questions in algebraic geometry. References to all these topics are given in the author's introduction. Buildings of affine type are not considered in these notes, and are the subject of another major work by Bruhat and the author [3]. Perhaps the most important application of all so far has been the author's classification of finite groups with BN-pairs of rank $\geqq 3$, which will be discussed later in the review.

Along with his axiomatization of the buildings, the author is also responsible for the concept of a group with a BN-pair (or Tits system, in the terminology of Bourbaki [2]). The axioms for a BN-pair in a group G are precisely the group theoretical conditions which G must satisfy in order to have a suitable set of parabolic subgroups from which the author's procedure of constructing a complex (outlined above in the case of $\mathrm{SL}_n(C)$) yields a building. The axioms for a BN-pair are also closely related to Chevalley's proof of the existence of a Bruhat decomposition in the Chevalley groups [5]. The building associated with a group G with a BN-pair admits G as a group of automorphisms, so that the group G is recovered from the building. Conversely the automorphism group of every building, and certain subgroups acting with sufficient transitivity, all have BN-pairs.

The problems whose solution are the main subject of the notes are as follows.

(A) The determination of the buildings of rank $\geqq 3$, and irreducible spherical type (which means having finite Weyl group). These all turn out to be associated with simple algebraic or classical groups. An application of (A) yields the enumeration of all finite groups with BN-pairs of irreducible type and rank $\geqq 3$, up to normal subgroups contained in B.

(B) The determination of all isomorphisms between buildings of rank $\geqq 2$ and spherical type, associated with algebraic or classical simple groups, and in particular, the determination of the full automorphism group of these buildings.

Problem A is a combinatorial analogue of the classification of algebraic simple groups of relative rank $\geqq 3$. The author remarks that the classification can easily be deduced from the solution of problem A. Details of this application are not included, but instead the reader is directed to what the author feels is a more efficient, if less elementary, method.

The notes begin with four sections on the basic theory of buildings, with their apartments, galleries, etc., and the connections between buildings and groups with BN-pairs. This part of the notes is more or less self-contained, except for a few references to Bourbaki [2], and provides a clear and fascinating introduction to the geometrical and group theoretical foundations of the subject. In particular, the discussion of Coxeter complexes in §2, in which reflections and roots emerge from the idea of foldings of the underlying complex, provides a new combinatorial basis for the study of groups generated by reflections, and their root systems.

In §5, the existence of buildings of various types is proved. This section, and everything that depends upon it, is much less elementary than the earlier sections, and is based primarily on the author's joint work with Borel [1] on reductive algebraic groups. This section concludes with the first of a number of detailed discussions of special cases and examples, devoted in this case to the connection between the projective plane over a Cayley division ring and buildings of type E_6.

The next section contains the classification of buildings of type A_n, D_n and E_n (which appear together naturally because their root systems are all amalgamations, in a certain sense, of root systems of type A_2). Buildings of type A_2 are the Desarguesian projective planes over division rings, and are shown to yield, by an amalgamating process, all buildings of these types.

The longest part of the notes is devoted to buildings of type C_n. The first section shows that buildings of type C_r are equivalent to polar spaces, studied by Veldkamp. The next section is devoted to conditions which permit polar spaces to be embedded in projective spaces by means of a polarity, and thereby classified. One of the author's main results in this section is an extension of Veldkamp's embedding theorem. This section includes a detailed and self-contained presentation of a generalization of the theory of quadratic forms needed to explain the embedding process.

Finally, in §9, nonembeddable polar spaces are classified. The buildings in this case are constructed from projective planes over Cayley division algebras. The author points out, without giving full details, the existence of another approach to this topic using the theory of reduced exceptional Jordan algebras, following Freudenthal.

The last buildings to be classified are those of type F_4. Many of the preceding results are needed to complete this enterprise.

In order for the classification of the buildings (problem A) to accomplish the original purpose—the geometrical interpretation of the semisimple groups—the groups themselves must appear either as the full automorphism groups or as easily recognizable subgroups of the automorphism groups of the buildings associated with them. This is the significance of problem B. In the case of projective spaces, the main result on problem B can be viewed as a version of the fundamental theorem of projective geometry.

In §11, the author works out what is at present perhaps the most important application of the solution of problem A—the classification of finite groups with BN-pairs of rank $\geqq 3$. This result, which had been announced by the author in 1961, has been used with considerable success to complete the proofs of certain characterizations of finite simple groups of Lie type. The method consists in showing that finite simple groups with certain properties (involving, for example, centralizers of involutions or Sylow subgroups) have a BN-pair of rank $\geqq 3$, and therefore must be of a known type because of the author's classification theorem.

Two appendices are included. The first, on shadows, is an analysis of the procedure used at various points in the notes, to associate spaces (projective spaces, polar spaces, etc.) with buildings. The second, on generators and relations, is the counterpart, for groups with BN-pairs, of a result in §4, which can be viewed as a kind of presentation of an arbitrary building of spherical type by amalgamation of buildings of rank one in buildings of rank two. In this appendix, it is shown that an arbitrary group G with a BN-pair is an amalgamated sum of the parabolic subgroups of rank 2 containing B. It is also shown that G can be characterized by means of the amalgamation of B in parabolic subgroups of rank one containing it.

The organization of the notes permits some sections to be used as introductions to some topics in algebra and geometry, without becoming involved with the overall plan for the solution of problems A and B. For example, the first four sections provide an introduction to buildings, groups with BN-pairs, and the combinatorial foundations of Coxeter groups. §7 provides an introduction to polar spaces, and the connection with buildings of type C_n. §8 introduces the theory of pseudo-quadratic forms, which have been used in differential topology. The geometry of conformal spaces and of projective Cayley planes is developed in §9. Connections with Coxeter's work also receive brief indications on p. 24 where realizations of Coxeter complexes as barycentric subdivisions of regular polytopes are described, and on p. 105, where thin polar spaces are related to the sets of vertices of cross polytopes. The two appendices also deserve mention here, as being of interest apart from the main task of solving problems A and B.

An extensive bibliography is included, and frequent references to it occur in the text. In particular some historical notes, with references, are given in the introduction.

There is no point for me to hide my enthusiasm for this work. It remains to express the hope that these notes will become familiar to a wide circle of readers, who will profit as much from the author's imagination, insight, and clear style, as I have.

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CHARLES W. CURTIS

Map color theorem, by Gerhard Ringel, Springer-Verlag, New York, Heidelberg, Berlin, 1974, 191+xii pp., $22.20

The four color conjecture is a famous problem that has challenged and stimulated mathematicians for more than a century. As most mathematicians know, it consists of the statement that with four colors one can color any map on a sphere such that any two countries with a boundary edge in common are of different colors. The present volume concerns a related problem: how many colors are necessary to color all similarly colored maps on surfaces of higher genus?

This problem has an entirely different flavor, as we shall see; it has a long history as well. It was posed by Heawood, who thought he had proven his conjectured answer in 1890. The last case was solved in 1968 (most cases solved by the author) verifying the original conjecture. A complete description in remarkably clear language of solutions for all cases is presented in this volume, which is written at a level suitable for an undergraduate seminar.

The major difference between the sphere problem and higher genus-surface problems is this: On a sphere one knows that one cannot have five countries every pair of which are neighbors—a configuration obviously requiring five colors—but one does not know if there is some large