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RICCATI SYSTEMS[1]

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Consider the differential equations

$$y^{(n)} + p(t)y = 0 \tag{1}$$

and

$$y^{(n)} - p(t)y = 0, \tag{2}$$

where p is positive and continuous on $[0, \infty)$, and suppose that, for some $a \geqslant 0$ and some integer $k \in [1, n-1]$ either of these equations has at least one nontrivial solution y for which

$$y(a) = y'(a) = \cdots = y^{(k-1)}(a) = y^{(k)}(c)$$

$$= y^{(k+1)}(c) = \cdots = y^{(n-1)}(c), \qquad c > a.$$

The point $\eta_{k,n-k}(a) = \inf c$, where c ranges over all values for which such solutions exist, is called the "$(k, n-k)$-focal point of a" (the fact that $\eta_{k,n-k}(a) > a$ is elementary). The point $\eta(a) = \min_k \eta_{k,n-k}(a)$ is referred to as "the focal point of a". It is known that equation (1) can only have focal points $\eta_{k,n-k}(a)$ for which $n-k$ is an odd number, while in the case of equation (2) $n-k$ must be even [4]. For the study of focal points we may therefore replace (1) and (2) by the single equation

$$y^{(n)} - (-1)^{n-k} py = 0. \tag{3}$$

In the oscillation theory of equations of the form (1) or (2), focal points play a role very similar to that of the more commonly used conjugate points [1]. In particular, it can be shown that the nonexistence of a focal point $\eta(a) \in (a, \infty)$ is equivalent to the disconjugacy of the equation on $[a, \infty)$.

Our principal result characterizes the focal points of an equation in terms of continuity properties of the solutions of an associated nonlinear differential system.

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THEOREM I. *Let C and c_1 denote, respectively, the $(n-k)\times(n-k)$ matrix*

$$C=\begin{pmatrix}0 & 1 & & 0\\ \cdot & \ddots & \ddots & \\ \cdot & & \ddots & 1\\ 0 & \cdot & \cdot & 0\end{pmatrix}$$

and the $(n-k)$-dimensional row vector $c_1=(1,0,\ldots,0)$. For the non-existence on an interval $[a, b)$ of a focal point $\eta_{k,n-k}(a)$ of equation (3) *it is necessary and sufficient that the solution of the differential system*

$$(4)\qquad S'_\mu = S_{\mu+1} + S_\mu C + pS_\mu^{(n-k)}S_1,\qquad \mu=1,\ldots,k,\quad S_\mu(a)=0,$$

$(S_{k+1}=c_1)$ for the row vectors $S_\mu=(S_\mu^{(1)},\ldots,S_\mu^{(n-k)})$ be continuous on $[a, b)$.

The S_μ are expressed in terms of certain combinations of solutions of equation (3). For instance, if $k=n-1$ (the simplest case), we set $S_\mu = y^{(\mu-1)}/y^{(n-1)}$, where y is the solution of (1) determined by the initial conditions $y^{(\nu)}(a)=0$, $\nu=0,\ldots,n-2$, $y^{(n-1)}(a)=1$. A simple computation then shows that the (scalar) functions S_μ satisfy the differential system

$$S'_\mu = S_{\mu+1} + pS_1S_\mu,\qquad \mu=1,\ldots,n-1,\quad S_\mu(0)=0,\quad S_n=1.$$

For $n=2$, this reduces to the Riccati equation $S' = 1 + pS^2$ $(S_1 = S)$ with the initial condition $S(a)=0$.

The usefulness of the "Riccati system" (4) is largely due to the fact that the coefficients of the equivalent system of scalar differential equations (for the components of the S_μ) are all nonnegative. This property can be utilized to obtain comparison theorems and criteria for the existence of focal points. The following are two examples of criteria obtainable in this way.

THEOREM II. *Equation* (3) *does not have a focal point $\eta_{k,n-k}(a)$ on $[a,\infty)$ if there exists a function $R(t)$ with the following properties: $R\in C^k$ and $R>0$ on $[a,\infty)$; $R^{(\nu)}(a)\geqslant 0$, $\nu=1,\ldots,k-1$; for all $t>a$, R satisfies the inequality*

$$\frac{1}{(n-k-1)!}\int_t^\infty s^{n-k-1}pR\,ds \leqslant R^{(k)}(t).$$

The function $R(t)=t^{k-\nu}$, $\nu<1$ has all the required properties, provided

$$(5)\qquad t^\nu\int_t^\infty p(s)s^{n-1-\nu}\,ds \leqslant (n-k-1)!\prod_{m=1}^{k}(m-\nu).$$

This condition thus guarantees the absence of a focal point $\eta_{k,n-k}(a)$ on $[a, \infty)$ $(a \geqslant 0)$. If $M(\nu)$ is the minimum for $k \in [1, n-1]$ of the right-hand side of (5), equations (1) and (2) will be disfocal, and therefore also disconjugate, on $[a, \infty)$ if p satisfies the condition

$$t^{\nu} \int_t^{\infty} p(s) s^{n-1-\nu}\, ds \leqslant M(\nu), \qquad t \in [a, \infty) \tag{6}$$

for some $\nu < 1$.

THEOREM III. *If equation* (3) *does not have a focal point* $\eta_{n,n-k}(a)$ *on* (a, ∞), *then*

$$t^{\nu} \int_t^{\infty} p(s) s^{n-1-\nu}\, ds \leqslant \nu^{-1}(n-1)^2 (k-1)!\,(n-k-1)! \tag{7}$$

for all $\nu \in (0, n-1)$ *and all* $t \in [a, \infty)$.

It may be noted that for $\nu \in (0, 1)$ the sufficient condition (6) and the necessary condition (7) differ only in the value of the constant on the right-hand side.

We finally remark that from (6) we can obtain disconjugacy criteria for equations $y^{(n)} + q(t)y = 0$ whose coefficients are not of constant sign. Since $-|q| \leqslant q \leqslant |q|$, and the disconjugacy of $y^{(n)} + q_1 y = 0$ and $y^{(n)} + q_2 y = 0$ implies that of $y^{(n)} + qy = 0$ if $q_1 \leqslant q \leqslant q_2$ [3], we find that $y^{(n)} + qy = 0$ is disconjugate on $[0, \infty)$ if

$$t^{\nu} \int_t^{\infty} |q| s^{n-1-\nu}\, ds \leqslant M(\nu), \qquad t \in [0, \infty)$$

for some $\nu < 1$. In particular, this contains the sufficient disconjugacy condition

$$\int_0^{\infty} |q| s^{n-1} ds \leqslant (n-2)! \qquad \text{(cf. [2])}.$$

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