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MANIFOLDS WITH PREASSIGNED CURVATURE—A SURVEY

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In this paper I discuss two problems of Riemannian geometry in the large concerning the existence of manifolds with preassigned curvature.

The Minkowski problem and its generalization asks in Euclidean space for a closed convex hypersurface whose curvature has been given in advance. The converse to the Gauss-Bonnet theorem asks for the existence, on a two-dimensional manifold, of a Riemannian metric with prescribed Gaussian curvature. The questions have a meeting point: the search for two-spheres in three-space with given strictly positive curvature.

While the first problem goes back to the work of Minkowski [32] in 1897, the second is of more recent vintage: it was posed explicitly by Warner in the early 1960's. Both have been solved in the last few years, and in this survey I try to give an overview and some of the details.

The paper is organized into the following sections:

1. The Minkowski problem
2. The generalized Minkowski problem
3. Converse to the Gauss-Bonnet theorem for smooth manifolds
4. Converse to the Gauss-Bonnet theorem for PL manifolds
5. Realization in three-space

1. The Minkowski problem.

(1.1) CURVATURE OF CONVEX HYPERSURFACES. Let M^n be a smooth closed convex hypersurface in Euclidean space R^{n+1}. The Gauss map $\gamma: M^n \to S^n$ associates with each point $x \in M$ the unit outward normal vector to M at x. Given a region A on M, the ratio

$$\frac{\text{area of } \gamma(A) \text{ on } S^n}{\text{area of } A \text{ on } M^n}$$

represents the average curvature of M throughout the region A, and its

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limit, as A shrinks down to x, is the curvature of M at x. For $n \geqq 3$ this is often called the *Gauss-Kronecker curvature* to distinguish it from other types of curvature.

If, in addition, M is strictly convex, then γ is a homeomorphism and its inverse is a parametrization of M. Under what conditions, Minkowski wanted to know, can one preassign a strictly positive function $K: S^n \to R^1$ and then find M^n as above so that its curvature at the point $\gamma^{-1}(p)$ is precisely the preassigned value $K(p)$ for all $p \in S^n$?

From a slightly different view, suppose one wanted to preassign the average curvature of the set $\gamma^{-1}(B)$ for each reasonable subset $B \subset S^n$. It would be enough to preassign the area of $\gamma^{-1}(B)$ on M, since then the average curvature would be the value of the fraction

$$\frac{\text{area of } B \text{ on } S^n}{\text{area of } \gamma^{-1}(B) \text{ on } M^n}.$$

Minkowski therefore recognized that preassigning an area function in the above sense had the same spirit as preassigning a curvature function.

He began by formulating the problem in the PL category.

(1.2) PL MINKOWSKI THEOREM. *Let $\vec{n}_1, \cdots, \vec{n}_m$ be distinct, noncoplanar unit vectors in R^3, and $F_1, \cdots, F_m$ real numbers >0 such that $\sum_{i=1}^n F_i\vec{n}_i = 0$. Then there exists in R^3 a closed convex polyhedral surface, unique up to translation, for which the $\vec{n}_i$ and F_i are the unit outer normals and surface areas of its faces.*

The condition $\sum_{i=1}^n F_i\vec{n}_i = 0$ can be thought of as expressing the fact that any such surface, when projected on any plane, must have an image with total area (counting algebraic signs) zero. Minkowski **[32]** gave a nice proof of this theorem, and Bonnesen and Fenchel **[9]** later recognized that with inessential changes it could be generalized to the case of convex hypersurfaces in a Euclidean space of any dimension. A. D. Alexandrov reproved this result in a striking way **[6]**, and since his methodology has far-reaching applications, I will sketch it instead.

Alexandrov began by proving the following geometric fact.

(1.3) LEMMA. *Let P_1 and P_2 be convex polygons in the plane which* ***cannot*** *be moved by parallel translation so that one contains the other. Then the difference of lengths of corresponding parallel sides must change sign at least four times as one circles P_1 or P_2.*

The convention in effect here is that any two convex polygons may be regarded as having pairwise parallel sides, some of which may have length 0. Thus, comparing a diamond with a square, each has eight sides, four of which are of length 0. The two figures alternate as to which has the larger corresponding side, so that the sign mentioned in the statement of the lemma changes eight times.

Next Alexandrov restated the following result of Cauchy.

(1.4) CAUCHY LEMMA. *Let $L \subset S^2$ be a finite graph having no circular edges and having no region of S^2-L bounded by just two edges. Mark the edges of L randomly with $+$ or $-$. Let N_v denote the number of changes of sign, as one circles v, of the edges touching that vertex. Let $N=\sum_v N_v$ denote the total number of sign changes, and V the number of vertices of L. Then*

$$N \leqq 4V - 8.$$

In particular, it is ***impossible*** *that each $N_v \geqq 4$.*

This lemma was proved and used by Cauchy **[12]** in the course of showing that all convex polyhedra in R^3 are rigid, and is done by a simple combinatorial argument, which the reader can find in **[6]** or **[31]**.

Next Alexandrov tackled the uniqueness part of the proof.

(1.5) UNIQUENESS. *Two convex polyhedra in R^3, whose faces are pairwise parallel and of equal area, differ at most by a parallel translation.*

To separate out the essential part of this argument, pretend that each pair of parallel faces has pairwise parallel edges. Pick one of the polyhedra, say P_1, and mark its edges $+$ or $-$, according as they are longer or shorter than the corresponding edges of P_2, and leave the edge unmarked if equal. Circling around the boundary of a face of P_1, we get no marked edges if that face is congruent via parallel translation to the corresponding face of P_2. Otherwise, since they have equal area, neither can be moved by parallel translation so as to contain the other, and the previous lemma then promises at least four sign changes around the boundary.

The spherical image P_1^* of P_1 under the Gauss map is combinatorially dual to it, and one then applies the Cauchy Lemma to P_1^*, first deleting the images of unmarked edges. The simple hypotheses of the Cauchy Lemma are easily seen to be satisfied. The conclusion is that no edge of P_1 was marked, and therefore that each face of P_1 is congruent via parallel translation to the corresponding face of P_2. It follows immediately that P_1 and P_2 are themselves congruent via parallel translation.

The pretense of pairwise parallel edges is avoided in a strict proof by introducing the polyhedron $P=(P_1+P_2)/2$ as a bookkeeping device. Its virtue lies in the fact that P^* is the greatest common subdivision of P_1^* and P_2^*. Details may be found in **[6]**.

(1.6) REMARK. Unfortunately, the above uniqueness argument does not work in higher dimensions. To see this, note that what was essentially proved is that if two convex polyhedra in R^3 have pairwise parallel faces, neither properly embeddable in the other via parallel translation, then these polyhedra must in fact be congruent via parallel translation. But this is already false in four-space, as the reader may see by comparing the hypercube of side 2 with a rectangular parallelepiped of sides 1, 1, 3, 3. Nevertheless, the uniqueness theorem is true in all dimensions via a different proof, namely Minkowski's.

(1.7) EXISTENCE. It is the existence part of Alexandrov's argument that is so striking. Let P be a convex polyhedron in R^3. The distance from the origin to the plane containing the ith face of P is denoted by h_i, with a positive sign if the origin and the rest of P are on the same side of this plane, and a negative sign if not. The numbers $h_1, \cdots, h_m$ are called the *support numbers* of P. If the outer normals $\vec{n}_1, \cdots, \vec{n}_m$ to the faces of P are known, then P may be reconstructed once its support numbers are known.

Return to the statement of the PL Minkowski theorem. We are told the (noncoplanar) unit outer normals $\vec{n}_1, \cdots, \vec{n}_m$ and the face areas $F_1, \cdots, F_m$ of a proposed convex polyhedron in R^3, subject to the relation $\sum_i F_i\vec{n}_i=0$, and must demonstrate the existence of such a polyhedron.

Keep $\vec{n}_1, \cdots, \vec{n}_m$ fixed but let $F_1, \cdots, F_m$ vary, still subject to the relation. Then we get a whole family of problems, that is, a *problem space*

$$\mathscr{P} = \left\{(F_1, \cdots, F_m): \quad \text{each } F_i > 0 \quad \text{and} \quad \sum_i F_i\vec{n}_i = 0\right\},$$

which can be viewed as an open convex set in some R^{m-3}. It is nonempty because there is certainly a convex polyhedron circumscribed about the unit sphere and having unit outer normals $\vec{n}_1, \cdots, \vec{n}_m$ to its faces.

By contrast, the solution space should simply be the set of all convex polyhedra in R^3 with these face normals. They may be parametrized by their support numbers $(h_1, \cdots, h_m)$. Conversely, given such support numbers, we may draw in the planes $\vec{n}_i \cdot \vec{x}_i=h_i$ and expect to see P emerge as the intersection of the corresponding half spaces $\vec{n}_i \cdot \vec{x}_i \leqq h_i$. Of course it may happen that some of the faces of P will not appear, since the intersection may equal an intersection with fewer terms. In any case, it is clear that the polyhedra P actually exhibiting all m faces correspond to an open subset $\mathscr{S}_0$ of the parameter space $R^m=\{(h_1, \cdots, h_m)\}$. If one now reckons two polyhedra equivalent when congruent via a parallel translation, and views this on the parameter space $\mathscr{S}_0$, one gets a quotient space

$$\mathscr{S} = \mathscr{S}_0/\text{parallel translation},$$

which is easily seen to correspond to an open subset of R^{m-3}. $\mathscr{S}$ is the *solution space* for the given family of problems, also nonempty by circumscription about the unit sphere in R^3.

What one wants to show is that each problem has a solution. What is in any case clear, Alexandrov points out, is that each solution has a problem. This is expressed by a map $\varphi:\mathscr{S}\to\mathscr{P}$, which is easily seen to be continuous. The uniqueness half of the proof, already settled, indicates that φ is one-to-one. Since $\mathscr{S}$ and $\mathscr{P}$ are both manifolds of dimension $m-3$, "invariance of domain" implies that φ is an open map.

To see that $\varphi(\mathscr{S})$ is a closed set is easy. But then $\varphi(\mathscr{S})$, as a nonempty open and closed subset of the connected space $\mathscr{P}$, must be all of $\mathscr{P}$. Thus every problem has a solution, which finishes the existence part of Alexandrov's argument, and with it his proof of the PL Minkowski theorem.

Using the PL version just proved, together with a convergence argument, Minkowski was then able to obtain the following more general result. We incorporate Bonnesen and Fenchel's observation [9] that Minkowski's original proof is easily extended to any dimension, by so stating the theorem.

(1.8) MINKOWSKI'S THEOREM. *Let F be a nonnegative, completely additive set function defined on all Borel subsets of the unit sphere $S^n\subset R^{n+1}$, $n\geqq 2$, such that*

(1) $\int_{S^n}\vec{n}F(d\Omega)=0$,

(2) $F(S)<F(S^n)$ *for every great $(n-1)$ sphere S on S^n.*

Then F is the area function of a closed convex hypersurface $M^n\subset R^{n+1}$ which is unique up to parallel translation.

Note that condition (1) is the analogue of the equation $\sum_i F_i\vec{n}_i=0$ in the PL version, while (2) is the analogue of the requirement that the vectors $\vec{n}_1,\cdots,\vec{n}_m$ be noncoplanar. See [33].

What Minkowski really wanted to prove, but did not, was

(1.9) SMOOTH MINKOWSKI THEOREM. *Let $K:S^n\to R^1$, $n\geq 2$, be a smooth strictly positive function such that $\int_{S^n}(\vec{n}(p)/K(p))d\Omega=0$. Then there exists a smooth closed convex hypersurface $M^n\subset R^{n+1}$, unique up to parallel translation, whose Gauss map $\gamma:M^n\to S^n$ is a homeomorphism, such that the Gaussian curvature of M at the point $\gamma^{-1}(p)$ is $K(p)$ for all $p\in S^n$.*

The most difficult thing was to prove the smoothness of the solution, given a smooth preassigned curvature function K, and this was done as follows.

$n=2$:

(1)	$K \in C^0 \Rightarrow M^2 \in C^1$	Alexandrov 1942 [4]
(2)	$K \in C^r \Rightarrow M^2 \in C^{r+1}, \quad r \geqq 2$	Pogorelov 1952 [38]
		Nirenberg 1953 [37]
(3)	K analytic $\Rightarrow M^2$ analytic	Lewy 1938 [30]

$n \geqq 3$:

(4)	$K \in C^r \Rightarrow M^2 \in C^r, \quad r \geqq 3$	Pogorelov 1971 [39]
(5)	K analytic $\Rightarrow M^2$ analytic	Pogorelov 1971 [39]

2. **The generalized Minkowski problem.** A question more primitive than that answered by Minkowski's theorems is whether a preassigned strictly positive curvature function, unencumbered by any integrability condition, can be realized by some embedding of $S^n \rightarrow R^{n+1}$, not necessarily the inverse of the Gauss map.

(2.1) GENERALIZED MINKOWSKI THEOREM. *Let $K: S^n \rightarrow R^1$, $n \geq 2$, be a strictly positive function. Then there exists an embedding $G: S^n \rightarrow R^{n+1}$ onto a closed convex hypersurface M^n, whose Gaussian curvature at the point $G(p)$ is $K(p)$ for all $p \in S^n$, subject to the following smoothness conditions*:

For $n=2$: $K \in C^r \Rightarrow G \in C^{r+1}$ for $r=0$ or $r \geqq 2$.
For $n \geqq 3$: $K \in C^r \Rightarrow G \in C^r$ for $r \geqq 3$.

The proof can be found in [17]. For $n=1$, this theorem has a variation in which the required embedding G exists if and only if K has at least two maxima and two minima, and one obtains thereby a converse to the classical 4-vertex theorem of differential geometry. A simplified proof, covering this case only, appears in [18].

(2.2) A REDUCTION OF THE PROOF. Given the strictly positive function $K: S^n \rightarrow R^1$ as in the hypotheses, one should first check in spirit whether or not $\int_{S^n} (\vec{n}(p)/K(p))\, d\Omega \overset{?}{=} 0$. If this integral vanishes, just apply the Smooth Minkowski Theorem (1.9) to get the result. But in general the integral does not vanish; its value is some random vector in R^{n+1}.

Suppose for the moment that the problem is solvable, that is, that there is an embedding $G: S^n \rightarrow R^{n+1}$ with the curvature of the image M^n at the point $G(p)$ being $K(p)$. Then the curvature of the image at the point $\gamma^{-1}(q)$, where $\gamma: M^n \rightarrow S^n$ is the Gauss map, is $K(G^{-1}\gamma^{-1}(q))$. Hence by the integrability condition for (1.9)

$$\int_{S^n} \frac{\vec{n}(p)}{Kh^{-1}(p)}\, d\Omega = 0,$$

where h stands for the diffeomorphism γG of S^n onto itself.

Conversely, if a diffeomorphism h of S^n onto itself exists making the above integral zero, let M^n be the solution of the smooth Minkowski problem for the curvature function Kh^{-1}. Then define $G=\gamma^{-1}h$, and note that the curvature of M^n at the point $G(p)=\gamma^{-1}(h(p))$ is $Kh^{-1}(h(p))=K(p)$.

(2.3) CONCLUSION. To solve the generalized Minkowski problem, it is necessary and sufficient to find a diffeomorphism h of S^n onto itself such that the above integral vanishes.

(2.4) REMARK. More generally, the following is true.

Let $M^n \subset R^{n+1}$, $n \geqq 2$, be a smooth closed manifold, $\vec{n}(x)$ the unit outward normal vector to M at x, and $f: M \to R^1$ any continuous function. Then there exists a diffeomorphism h of M onto itself, diffeotopic to the identity, for which

$$\int_{M^n} fh^{-1}(x)\vec{n}(x)\, dA = 0.$$

This can also be found in [**17**].

(2.5) ANALOGY WITH A PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA. The hunt for such a diffeomorphism h proceeds by analogy with the topologist's hunt for a root of the complex polynomial equation $f(z)=0$. Here we are looking for a root of the equation

$$I(h) = \int_{M^n} fh^{-1}(x)\vec{n}(x)\, dA = 0.$$

One starts by regarding the group Diff(M^n) of diffeomorphisms of M onto itself with the C^∞ topology as the analogue of the complex plane, but soon abandons this in favor of a certain $(2n+2)$-cell B^{2n+2} in Diff(M^n), which now shares with the complex plane the advantage of being contractible. A certain n-sphere $\Sigma^n \subset B^{2n+2}$ is constructed to play a role analogous to that of a circle of large radius in the complex plane. Specifically, no roots of the equation $I(h)=0$ can be found on Σ^n. Thus

$$I|\Sigma^n : \Sigma^n \to R^{n+1} - \{0\},$$

and the final step is to prove, still faithful to the analogy, that this map is essential. The conclusion that I has roots in B^{2n+2} is then inescapable, and the argument ends.

3. **Converse to the Gauss-Bonnet theorem for smooth manifolds.** If the smooth closed two-manifold M^2 has a Riemannian metric with curvature K, then the beautiful theorem of Gauss [**15**] and Bonnet [**10**] relates this to the Euler characteristic:

$$\int_M K\, dA = 2\pi\chi(M).$$

This in turn limits the possible functions which can serve as the curvature of a Riemannian metric on a given surface. For example, a negative function on the two-sphere could not possibly be the Gaussian curvature of a Riemannian metric there because there is no way it could integrate to 4π. This constraint, imposed on the curvature by the topology of the surface, is all the more remarkable in that no others are known.

(3.1) THE SIGN CONDITION. If K is to be the Gaussian curvature of some Riemannian metric on M, we cannot formulate the Gauss-Bonnet theorem in advance, because before M gets a Riemannian metric, it does not have an area element. We can extract, however, the following constraint on the sign of K:

(a) If $\chi(M)>0$, K must be positive somewhere on M.

(b) If $\chi(M)=0$, K (if not identically 0) must change sign.

(c) If $\chi(M)<0$, K must be negative somewhere.

(3.2) THREE RELATED PROBLEMS. The converse to the Gauss-Bonnet theorem can now be recorded as

CONJECTURE A. *If K satisfies the sign condition, then there is a Riemannian metric on M having K as its Gaussian curvature.*

We see from the Generalized Minkowski Theorem (2.1) that this conjecture is true if M is a two-sphere and $K>0$ everywhere. In the remaining cases, Kazdan and Warner found it advantageous to relate the problem to two further conjectures.

CONJECTURE B. *If K satisfies the sign condition, then there is a Riemannian metric on M, conformally equivalent to a preassigned metric, having K as its Gaussian curvature.*

The metrics g and g_0 on M are *conformally equivalent* if there is a smooth real valued function u on M and a diffeomorphism φ of M onto itself such that $g=\varphi^*(e^{2u}g_0)$, the pullback of $e^{2u}g_0$.

CONJECTURE C. *If K satisfies the sign condition, then there is a Riemannian metric on M, pointwise conformal to a preassigned metric, having K as its Gaussian curvature.*

The metrics g and g_0 are *pointwise conformal* if there is a smooth real valued function u on M such that $g=e^{2u}g_0$.

Clearly, Conjecture C$\Rightarrow$Conjecture B$\Rightarrow$Conjecture A.

Here is a table of results.

Gluck [17]. Conjecture A is true on the two-sphere if $K>0$.

Moser [34]. Conjecture C is true on the projective plane.

Poincaré [40] and Mel Berger [8]. Conjecture C is true on surfaces of strictly negative Euler characteristic if $K<0$.

Kazdan and Warner [24]. Conjecture C is false, except on the projective plane.

Kazdan and Warner [24]. Conjecture B is true, except possibly for the two-sphere.

Kazdan and Warner [27], [28]. Conjecture B is true, even for the two-sphere.

CONCLUSION. Conjecture C is true for the projective plane and false otherwise.

Conjecture B is true.

Conjecture A, the converse to the Gauss-Bonnet theorem for smooth closed two-manifolds, is true.

(3.3) THE EQUATION $\Delta_0 u = K_0 - Ke^{2u}$. In order to hint at the details, we must first translate the condition $g = e^{2u}g_0$ of Conjecture C into a partial differential equation for the unknown function u on M. Let K_0 be the Gaussian curvature of the original metric g_0, and K the Gaussian curvature of the desired metric g.

To derive Equation (3.3) in which Δ_0 denotes the Laplace operator for the metric g_0, write

$$g_0 = e^{2v(x,y)}(dx^2 + dy^2)$$

in local conformal parameters [13]. Then one computes:

$$K_0 = e^{-2v}(v_{xx} + v_{yy}), \qquad \Delta_0 u = -e^{-2v}(u_{xx} + u_{yy}).$$

If we apply the above curvature formula to the metric

$$g = e^{2u}g_0 = e^{2(u+v)}(dx^2 + dy^2),$$

we get

$$K = e^{-2(u+v)}(u_{xx} + v_{xx} + u_{yy} + v_{yy}).$$

But then

$$K_0 - Ke^{2u} = -e^{-2v}(u_{xx} + u_{yy}) = \Delta_0 u,$$

as desired.

(3.4) INTEGRABILITY CONDITIONS FOR $\Delta_0 u = K_0 - Ke^{2u}$. Roughly, the plan is first to try to solve Equation (3.3) for the unknown function u, which amounts to answering Conjecture C affirmatively. Failing this, one tries for integrability conditions for the equation. If they are not satisfied for the proposed curvature function K, one then looks for an appropriate diffeomorphism $\varphi: M^2 \to M^2$ so that they will be satisfied for the curvature function $K\varphi^{-1}$. In that case, $K\varphi^{-1}$ will be the curvature of a metric $e^{2u}g_0$ on M^2, and hence K the curvature of the metric $\varphi^*(e^{2u}g_0)$, which will answer Conjecture B affirmatively.

Here are a few details.

(A) *On the two-sphere S^2.*

Koutroufiotis [29]. If K is antipodally symmetric, $K(x) = K(-x)$, and close in measure to $K_0 \equiv 1$, then Equation (3.3) is integrable.

Moser [34]. If $K(x) = K(-x)$ and $K_0 \equiv 1$, the equation is integrable.

Kazdan and Warner [24]. In general, Equation (3.3) is *not* integrable. For example, suppose g_0 is the metric induced on the unit sphere $S^2 \subset R^3$ by the Euclidean metric of R^3, so that $K_0 \equiv 1$. Then a necessary condition which K must satisfy in order that (3.3) be integrable is

$$\int_{S^2} e^{2u}(\nabla_0 K)\cdot(\nabla_0 F)\, dA_0 = 0,$$

for all first order spherical harmonics F, where $\nabla_0 K$ and $\nabla_0 F$ denote the gradient vector fields of the functions K and F with respect to the round metric g_0, the dot product is with respect to g_0, and dA_0 is the area element of g_0. Now the coordinate functions x, y, z are the first order spherical harmonics on S^2 in the standard metric. If $K=z+2$, for example, so that it is strictly positive, and if $F=z$, then the above integral is also strictly positive, so that the integrability condition is *not* satisfied. Hence $K=z+2$ *cannot* be the curvature of a metric on S^2 pointwise conformal to the standard metric g_0. Similar integrability conditions are obtained by Kazdan and Warner for any initial metric g_0.

(B) *On the projective plane P^2.*

Moser [**34**]. If $K_0\equiv 1$, then Equation (3.3) is always integrable on the projective plane, and an extension of Moser's techniques yields the same result for any initial metric g_0.

(C) *On the torus and Klein bottle.*

Kazdan and Warner [**24**]. Suppose g_0 is a flat metric, so that $K_0\equiv 0$. If K is identically 0, the problem is trivial; otherwise we know that K takes both positive and negative values by the sign condition. In this case, Kazdan and Warner obtain a *necessary and sufficient* condition for the integrability of (3.3), namely

$$\int_{M^2} K\, dA_0 < 0.$$

Note the hybrid nature of this condition. It looks suspiciously like the Gauss-Bonnet theorem, except that equality is replaced by inequality. The point is that K is the curvature of the desired metric $g=e^{2u}g_0$, while dA_0 is the area element of the original metric g_0.

This is a good place to see the general plan in action. If the above integrability condition is not satisfied by the proposed curvature function K, first find a point p in M^2 at which $K(p)<0$. Then find a diffeomorphism φ of M^2 onto itself which spreads a small neighborhood U of p, on which $K<0$, over most of M^2. Then $K\varphi^{-1}<0$ on most of M^2, so that the integrability condition $\int_{M^2} K\varphi^{-1}\, dA_0<0$ is certainly satisfied. Thus $K\varphi^{-1}$ is the curvature of a metric $e^{2u}g_0$ on M^2, and, hence, K is the curvature of the metric $\varphi^*(e^{2u}g_0)$. Therefore Conjecture B is answered affirmatively in this case. Similar results are obtained for any initial metric g_0.

(D) *On surfaces with $\chi<0$.*

Poincaré [40] and (by a different method) Mel Berger [8]. If $K<0$ everywhere on M^2, then Equation (3.3) is integrable for any initial metric g_0.

Kazdan and Warner [24]. Suppose g_0 is a metric of constant negative Gaussian curvature. Then a necessary condition for the solvability of (3.3) is that

$$\int_{M_2} K\, dA_0 < 0,$$

just as for the torus and Klein bottle, *except* that now the condition is no longer sufficient. Similar results hold for any initial metric g_0.

(3.5) It is clear from the above remarks that the treatment of Equation (3.3), and hence of Conjectures B and C, depends heavily on the topology of the surface M^2. Here is one more item in that direction.

Rewrite Equation (3.3) as

$$T(u) = -e^{-2u}(\Delta_0 u - K_0) = K,$$

and note that $T(0)=K_0$. It is natural to linearize this map T, because if the linearized version is invertible, then one can apply the inverse function theorem for Banach spaces to solve Equation (3.3) for K sufficiently close to K_0. The curious result is that the linearized version is invertible for the projective plane with the standard metric and for surfaces of strictly negative Euler characteristic, but not invertible for certain initial metrics on the two-sphere, torus and Klein bottle (Kazdan and Warner [27], Koutroufiotis [29]).

(3.6) For a uniform overview of the attack on Conjectures B and C, I suggest Kazdan and Warner [27]. For a somewhat different approach to Conjecture A, without the intermediacy of Conjectures B and C, see Kazdan and Warner [28], which depends on a result of Bourguignon [11] and Fisher and Marsden [14].

(3.7) NONCOMPACT SURFACES. *Let M^2 be a noncompact surface, diffeomorphic to an open subset of some compact surface. Then any C^∞ function is the Gaussian curvature of some Riemannian metric on M^2.*

See Kazdan and Warner [25].

4. Converse to the Gauss-Bonnet theorem for PL manifolds. Now let M^2 be a compact PL two-manifold, possibly with boundary. A PL Riemannian metric on M is determined by a triangulation of M and by linear metrics on each two-simplex of the triangulation, consistent with respect to the face operation. The *curvature* at an interior vertex is 2π minus the sum of the angles around that vertex; the *exterior angle* at a boundary vertex is π minus the angle sum.

The elementary geometry of Riemannian polyhedra unfolds in a manner similar to that for smooth manifolds. For details in this direction, including a discussion of curvature in the PL category, I suggest [**6**], [**7**], [**19**] and [**42**].

In any case, the Gauss-Bonnet theorem holds for compact PL surfaces, and reads:

$$\sum_{\mathring{M}} \text{curvature} + \sum_{\partial M} \text{exterior angles} = 2\pi\chi(M).$$

Since curvature here is the analogue of integral curvature in the smooth category, this equation acts, undiluted, as a constraint on any such preassigned data.

(4.1) THEOREM (JOINT WITH KENNETH KRIGELMAN AND DAVID SINGER). *Let M be a connected compact PL two-manifold possibly with boundary. Let $p_1, \cdots, p_r$ be points in the interior $\mathring{M}$ of M. Let $q_1, \cdots, q_s$ be points on the boundary ∂M of M.*

Given the data: (1) *real numbers $k_1, \cdots, k_r$, each $<2\pi$,*

(2) *real numbers $e_1, \cdots, e_s$, each $<\pi$, subject to the constraint:*

$$\sum_i k_i + \sum_j e_j = 2\pi\chi(M).$$

Then there exists a PL Riemannian metric on M with curvatures k_i at each p_i, exterior angles e_j at each q_j, and flat elsewhere.

The proof is combinatorial, somewhat reminiscent of the classification procedure for compact triangulable surfaces. It was first obtained by Singer for the two-sphere with positive preassigned curvature, then for the two-sphere without this restriction by the author, then by Krigelman (Ph.D. thesis, University of Pennsylvania, 1972) for closed surfaces, and finally in the above form. The reader is referred to [**19**] for details.

5. **Realization in three-space.** In both the smooth and PL versions of the converse to the Gauss-Bonnet theorem, the desired Riemannian metric was produced in abstracto, not via any specific model in Euclidean space. One can of course add this as an extra requirement, and again ask for the answer.

In the smooth case there is no problem because, by the Nash embedding theorem [**36**], a Riemannian metric on a compact manifold can always be realized by a model in Euclidean space.

In the PL case, the analogue of the Nash embedding theorem is yet to be proved. Even the specific Riemannian metrics obtained in [**19**] to exhibit preassigned curvature data are not known to be realizable in Euclidean space.

Consider, by contrast, the question of realizability of solutions in 3-space. In general, of course, the answer is no. A smooth closed surface in 3-space must have points of strictly positive curvature. Hence a metric with Gaussian curvature everywhere negative on a closed surface of negative Euler characteristic cannot be realized in 3-space. And the closed nonorientable surfaces cannot even be realized in 3-space for topological reasons. Similarly in the PL category.

One case is a natural for the extra requirement of embeddability in 3-space, namely that of positive preassigned curvature on a manifold homeomorphic to the two-sphere. In the smooth case one gets this for free.

The *Weyl problem* starts with a Riemannian metric of strictly positive curvature on a smooth manifold homeomorphic to the two-sphere, and asks for a realization by a smooth (necessarily convex) surface in 3-space. This problem was posed and partially solved by Weyl in 1916 **[43]**, complete solutions being given by Pogorelov **[38]** and Nirenberg **[37]**. As a consequence, once a strictly positive curvature function on a space homeomorphic to the two-sphere is realized by some Riemannian manifold in abstracto, one just appeals to the solution of the Weyl problem to embed it isometrically in 3-space.

In the PL case, the corresponding argument misses. The PL version of the Weyl problem starts with a PL Riemannian metric of positive curvature on a manifold homeomorphic to the two-sphere. One cannot possibly begin with strictly positive curvature, because all but finitely many points must have zero curvature. The affirmative solution was given by Alexandrov **[6]**. But it has one degenerate case. A doubly covered plane region, bounded by a convex polygon, is considered a degenerate case of a closed convex surface in 3-space. This degenerate case is unavoidable because it exists, and because the corresponding uniqueness result (the Cauchy rigidity theorem) applies to it and prevents an alternative nondegenerate realization.

Unfortunately, the solution given in **[19]** to the converse of the PL Gauss-Bonnet theorem lands one exactly on the degenerate case of the Weyl problem. The uniqueness aspect of the Weyl problem does not prevent an alternative realization as a nondegenerate closed convex surface in 3-space, since here we are interested in realizing only preassigned curvature, not preassigned metric.

An ingenious argument has been given by David Singer **[41]**.

(5.1) THEOREM (DAVID SINGER). *Let $p_1, \cdots, p_r$ be points on a PL two-sphere S and $k_1, \cdots, k_r$ real numbers with $0<k_i<2\pi$ and $\sum_i k_i=4\pi$. Then there exists a PL embedding of S into R^3 with convex image, realizing these preassigned curvatures k_i at the points p_i and flat elsewhere.*

(5.2) TETRAHEDRA WITH PREASSIGNED CURVATURE. In order to carry out the inductive step of Singer's argument, one needs to know the existence of tetrahedra with preassigned curvature and base.

Starting for example with a triangle $V_1V_2V_3$ in 3-space, one wants to extend it to a tetrahedron $V_1V_2V_3V_4$ with curvatures k_1, k_2, k_3, k_4 at its four vertices, where $0<k_i<2\pi$ and $\sum_i k_i=4\pi$. If the angles of $V_1V_2V_3$ are denoted by a_1, a_2, a_3, then one must assume $k_i<2\pi-2a_i$ or a solution will be impossible. Given this data, however, the desired tetrahedron exists and is unique up to reflection through its base.

The argument, though not difficult, reflects the charm of Alexandrov's methods. Parametrize the space of all tetrahedra with base $V_1V_2V_3$ by the numbers x_1, x_2, x_3 which represent the distance from V_4 to V_1, V_2, V_3, respectively. We are therefore not distinguishing between a tetrahedron and its reflection through the base. The curvatures k_1, k_2, k_3 depend differentiably on x_1, x_2, x_3 and one may compute the Jacobian determinant of this dependence. Somewhat involved at first, the computation yields the explicit answer:

$$\frac{\partial(k_1, k_2, k_3)}{\partial(x_1, x_2, x_3)} = 2\,\frac{\cos(\alpha + \beta + \gamma) - 1}{x_1x_2x_3 \sin \alpha \sin \beta \sin \gamma},$$

where α, β, γ are the angles at V_4 of the three faces. Since $0<\alpha+\beta+\gamma<2\pi$, this Jacobian is always <0. If one regards the set of parameters $\{(x_1, x_2, x_3)\}$ as the solution space $\mathscr{S}$, and the set of allowable curvatures $\{(k_1, k_2, k_3)\}$ as the problem space $\mathscr{P}$, then the natural map $\varphi:\mathscr{S}\rightarrow\mathscr{P}$ has nonsingular Jacobian everywhere. Both $\mathscr{S}$ and $\mathscr{P}$ are homeomorphic to open 3-cells, so φ must be an open map.

To see that φ is also a closed map, Singer argues as follows. Compactify the solution space $\mathscr{S}$ by adding "idealized solutions", essentially degenerate tetrahedra with control on the angles at each vertex, to form a space $\overline{\mathscr{S}}$ homeomorphic to a closed 3-cell. Similarly, compactify the problem space $\mathscr{P}$ by adding degenerate problems, that is, allowing $0\leq k_i\leq 2\pi-2a_i$, to form a space $\overline{\mathscr{P}}$ also homeomorphic to a closed 3-cell. The map φ extends to a map $\bar{\varphi}:\overline{\mathscr{S}}\rightarrow\overline{\mathscr{P}}$, which takes $\overline{\mathscr{S}}-\mathscr{S}$ to $\overline{\mathscr{P}}-\mathscr{P}$. In other words, degenerate solutions correspond to degenerate problems. But the map $\bar{\varphi}$ is closed because $\overline{\mathscr{S}}$ is compact. It follows immediately that φ is closed.

Now $\varphi(\mathscr{S})$ is a nonempty open and closed subset of the connected space $\mathscr{P}$, hence all of $\mathscr{P}$, and therefore every problem has a solution.

(5.3) REMAINDER OF SINGER'S PROOF. Using this lemma about tetrahedra, Singer constructs a surface bounding a polyhedral 3-cell in R^{r-1} and having preassigned strictly positive curvatures $k_1, \cdots, k_r$. The argument is by induction on r, so that the 3-cell is constructed by adding an appropriate tetrahedron to a polyhedral 3-cell in R^{r-2}, which was previously obtained as a solution to a corresponding problem with only $r-1$ preassigned curvatures. The step in dimensions from $r-2$ to $r-1$ is there to prevent self-intersections: if the construction is carried out in R^3, one cannot prove convexity directly by this method, and therefore cannot avoid self-intersections during the inductive step.

The final part of the argument is a surprise. One has just produced by the above method a polyhedral two-sphere with preassigned curvatures $k_1, \cdots, k_r$, but it lies in R^{r-1} rather than in R^3. Now comes an application of Alexandrov's solution of the Weyl problem to produce an isometric copy of this two-sphere as a convex polyhedral surface in 3-space. The problem is to avoid the degenerate case.

Notice that in a degenerate solution to the Weyl problem, namely a doubly covered plane convex polygonal region, each vertex of positive curvature can be joined to its immediate neighbor on either side by a unique path of shortest length on the surface. But to every other vertex of positive curvature it has exactly two shortest paths on the surface, one on each sheet.

Compare this with the two-sphere bounding the 3-cell in R^{r-1}, which in turn was obtained by adding a tetrahedron to a 3-cell in R^{r-2}. The vertex of that tetrahedron is a point of positive curvature on the resulting two-sphere, and it is joined by straight line paths in R^{r-1} to the three vertices at its base, all of which represent points of positive curvature on the two-sphere. Because the model is in Euclidean space, there can be no other paths of that same shortest length in the ambient space, let alone on the two-sphere.

It follows that this two-sphere cannot be isometric to any degenerate solution of the Weyl problem. But it is isometric to some solution because it has positive curvature, so that the solution must be nondegenerate and therefore furnishes the required convex two-sphere in 3-space with the preassigned curvature.

(5.4) Robert Connelly has found, by an entirely different method, an explicit construction of a convex polyhedral two-sphere circumscribed about the unit sphere in 3-space and having preassigned positive curvature.

References

1. A. D. Alexandrov, *Zur Theorie gemischter Volumina konvexer Körper*. I, II, Mat. Sb. **2** (1937), 947–972, 1205–1238.

2. ———, *Zur Theorie gemischter Volumina*. III, Mat. Sb. **3** (1938).

3. ———, *Über die Oberflächenfunktion eines konvexen Körpers*. (*Bemerkung zur Arbeit "Zur Theorie der gemischten Volumina von konvexen Körpern"*, Mat. Sb. **6 (48)** (1939), 167–174. (Russian) MR **1**, 265.

4. ———, *Smoothness of the convex surface of bounded Gaussian curvature*, C.R. (Dokl.) Acad. Sci. URSS **36** (1942), 195–199. MR **4**, 169.

5. ———, *Intrinsic geometry of convex surfaces*, OGIZ, Moscow, 1948; German transl., Akademie-Verlag, Berlin, 1955. MR **10**, 619; **17**, 74.

6. ———, *Convex polyhedra*, GITTL, Moscow, 1950; German transl., Akademie-Verlag, Berlin, 1958. MR **12**, 732; **19**, 1192.

7. T. F. Banchoff, *Critical points and curvature for embedded polyhedra*, J. Differential Geometry **1** (1967), 245–256. MR **37** #921.

8. Melvyn S. Berger, *Riemannian structures of prescribed Gaussian curvature for compact* 2-*manifolds*, J. Differential Geometry **5** (1971), 325–332. MR **45** #4329.

9. T. Bonnesen and W. Fenchel, *Theorie der Konvexen Körper*, Julius Springer, Berlin, 1934.

10. O. Bonnet, J. École Polytechnique **19** (1848), 1–146.

11. J.-P. Bourguignon, Thesis, Univ. of Paris VII, 1974.

12. A. Cauchy, *Sur les polygones et polyedres*, Second Memoire, J. École Polytechnique **9** (1813), 87.

13. S.-S. Chern, *An elementary proof of the existence of isothermal parameters on a surface*, Proc. Amer. Math. Soc. **6** (1955), 771–782. MR **17**, 657.

14. A. Fisher and J. Marsden, *The manifold of Riemannian metrics with prescribed scalar curvature* (to appear).

15. C. F. Gauss, *Disquisitiones generales circa superficies curvas*, 1827; English translation: *General investigations on curved surfaces*, Princeton, 1902.

16. H. Gluck, *Deformations of normal vector fields and the generalized Minkowski problem*, Bull. Amer. Math. Soc. **77** (1971), 1106–1110. MR **44** #2179.

17. ———, *The generalized Minkowski problem in differential geometry in the large*, Ann. of Math. (2) **96** (1972), 245–276. MR **46** #8132.

18. ———, *The converse to the Four-Vertex Theorem*, Enseignement Math. **17** (1971), 295–309.

19. H. Gluck, K. Krigelman and D. Singer, *The converse to the Gauss-Bonnet Theorem in* PL, J. Differential Geometry **9** (1974), 601–616.

20. J. L. Kazdan and F. W. Warner, *Integrability conditions for* $\Delta u=k-Ke^{au}$ *with applications to Riemannian geometry*, Bull. Amer. Math. Soc. **77** (1971), 819–823. MR **43** #8026.

21. ———, *Curvature functions for* 2-*manifolds with negative Euler characteristic*, Bull. Amer. Math. Soc. **78** (1972), 570–574.

22. ———, *Curvature functions for* 2-*manifolds*, Proc. Sympos. Pure Math. vol. 23, Amer. Math. Soc., Providence, R.I., 1973, pp. 387–392.

23. ———, *Surfaces of revolution with monotonic increasing curvature, and an application to the equation* $\Delta u=1-Ke^{2u}$ *on* S^2, Proc. Amer. Math. Soc. **32** (1972), 139–141.

24. ———, *Curvature functions for compact* 2-*manifolds*, Ann. of Math. (2) **99** (1974), 14–47.

25. ———, *Curvature functions for open* 2-*manifolds*, Ann. of Math. (2) **99** (1974), 203–219.

26. ———, *Scalar curvature and conformal deformation of Riemannian structure*, J. Differential Geometry (to appear).

27. ———, *Existence and conformal deformation of metrics with prescribed Gaussian scalar curvatures*, Ann. of Math. (to appear).

28. ———, *A direct approach to the determination of Gaussian and scalar curvature functions* (to appear).

29. D. Koutroufiotis, *On Gaussian curvature and conformal mapping*, J. Differential Geometry **7** (1972), 481–490.

30. H. Lewy, *On the existence of a closed surface realizing a given Riemannian metric*, Proc. Nat. Acad. Sci. U.S.A. **24** (1938), 104–106.

31. L. A. Ljusternik, *Convex figures and polyhedra*, GITTL, Moscow, 1956; English transl., Dover, New York, 1963. MR **19**, 57; **28** #4427.

32. H. Minkowski, *Allgemeine Lehrsätze über konvexe Polyeder*, Nachr. Ges. Wiss., Göttingen 1897, 198–219.

33. ———, *Volumen und Oberflache*, Math. Ann. **57** (1903), 447–495.

34. J. Moser, *On a nonlinear problem in differential geometry*, Dynamical Systems, Academic Press, New York, 1973, pp. 273–280.

35. ———, *A sharp form of an inequality of N. Trudinger*, Indiana Univ. Math. J. **20** (1970/71), 1077–1092. MR **46** #662.

36. J. F. Nash, Jr., *The imbedding problem for Riemannian manifolds*, Ann. of Math. (2) **63** (1956), 20–63. MR **17**, 782.

37. L. Nirenberg, *The Weyl and Minkowski Problems in differential geometry in the large*, Comm. Pure Appl. Math. **6** (1953), 337–394. MR **15**, 347.

38. A. V. Pogorelov, *Regularity of a convex surface with given Gaussian curvature*, Mat. Sb. **31** (**73**) (1952), 88–103. (Russian) MR **14**, 679.

39. ———, *On a regular solution of the n-dimensional Minkowski problem*, Dokl. Akad. Nauk SSSR **199** (1971), 785–788=Soviet Math. Dokl. **12** (1971), 1192–1196. MR **44** #2180.

40. H. Poincaré, *Les fonctions fuchsiennes et l'equation* $\Delta u = e^u$, J. Math. (5) **4** (1898), 137–230.

41. D. Singer, *Preassigning curvature on the two-sphere*, J. Differential Geometry **9** (1974), 633–638.

42. D. Stone, *Geodesics in PL manifolds* (to appear).

43. H. Weyl, *Über die bestimmung einer geschlossenen konvexen flache durch ihr linienelement*, Vierteljahrsschrift der naturforschen der Gesellschaft Zürich **61** (1916), 40–72.

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