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COMMUTATIVE SUBALGEBRA OF $L^1(G)$ ASSOCIATED WITH A SUBELLIPTIC OPERATOR ON A LIE GROUP G

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1. **Introduction.** Let G be a Lie group and LG its Lie algebra regarded as the space of differential operators of the first order which commute with the right translations. If $X_1, \cdots, X_n$ is a basis of LG, then the operator $L = X_1^2 + \cdots + X_n^2$ is called a laplacian on G. In [4] the commutative Banach $*$-subalgebra of $L^1(G)$ generated by the fundamental solution of the heat equation $(\partial/\partial t - L)u = 0$ was studied, and in case of compact extensions of nilpotent groups it proved to be useful in studying spectral properties of L on various $L^p(G)$ spaces, as well as in proving tauberian Wiener theorems concerning Gauss and Poisson integrals. In [6] and [9] a powerful method of singular integrals on the class of nilpotent Lie groups admitting one-parameter groups of dialations was developed. In [1] and [2] Folland and Stein studied the relation of these to certain subelliptic operators on the Heisenberg group. The idea is that in various important cases, although for a given one-parameter group of dialations $\{\delta_s\}, s > 0,$ of G there is no basis in LG such that $\delta_{s*}L = s^\lambda L$ where λ is a scalar, there exists a set of generators $X_1, \cdots, X_k$ of the Lie algebra LG such that $\delta_{s*}X_j = sX_j, j = 1, \cdots, k.$ Let

$$(1) \qquad L = X_1^2 + \cdots + X_k^2.$$

Then, of course,

$$(2) \qquad \delta_{s*}L = s^2 L.$$

The fact that $X_1, \cdots, X_k$ generate LG as a Lie algebra implies that L is a subelliptic operator. Using this fact we shall construct the Gauss and Poisson kernels for the operator L, and via a study of the subalgebra of $L^1(G)$ generated by these, we obtain the equality of the spectra of L on various $L^p(G)$ spaces as well as the corresponding tauberian Wiener theorems. More-

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over (2) implies very natural transformation rules for the Gauss and Poisson kernels under the dialations.

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2. **Theorems.** Consider L as a densly defined symmetric operator on $L^2(G)$. Let $L' = \mathbf{L}$ be its adjoint. Then, as it follows from [3], $\mathbf{L}$ is a selfadjoint nonpositive definite operator.

From [7] and [8] we deduce the following version of

SOBOLEV'S LEMMA. *There exists an* N *such that for every compact set* Ω *in* G *there exists a constant* C *such that* $|f(x)| \leqslant C\Sigma_{j=0}^{N}|\mathbf{L}^j f|_2$ *for all* x *in* Ω *and* f *in* $\mathcal{D}(\mathbf{L}^N)$.

Now we are able to define the Gauss and Poisson kernel pretty much the same way as in [4] and [11].

THEOREM 1. *There exists a unique one-parameter semigroup* $\{p_t\}_{t>0}$ *of* (i) *nonnegative,* (ii) *normalized functions in* $L^p(G)$ *such that*

(iii) $p_s * p_t = p_{s+t}$, $p_t^* = p_t$, *for every* f *in* $L^p(G)$,

(3) (iv) $\lim_{t\to 0}|p_t * f|_p = 0$, $1 \leqslant p < \infty$,

(v) *the function* $(0, \infty) \times G \ni t, x \longrightarrow p_t(x) \in \mathbf{R}$ *is* C^∞, *and if* $u(t, x) = p_t * f(x)$,

$$(d/dt - L)u(t, x) = 0 \tag{4}$$

for all $f \in L^p(G)$, $1 \leqslant p \leqslant \infty$.

Let

$$p^t(x) = (\pi)^{-1/2}\int_0^\infty \lambda^{-1/2}e^{-\lambda}p_{t^2/4\lambda}\,d\lambda. \tag{5}$$

Then $\{p^t\}_{t>0}$ *is a semigroup of functions in* $L^1(G)$ *such that* (i)–(iv) *are satisfied, and if* $u(t, x) = p^t * f$, *then*

$$(d^2/dt^2 + L)u(t, x) = 0 \tag{6}$$

for all $f \in L^p(G)$, $1 \leqslant p \leqslant \infty$.

Understandably enough, the function $u(t, x) = p_t * f(x)$ is called the Gauss integral of f, and the function $v(t, x) = P^t * f(x)$ is called the Poisson integral of f.

We have also the following version of Nelson's lemma [10].

THEOREM 2. *For every nonnegative submultiplicative function* ϕ *on* G *and every* $t_0 > 0$, *there is a constant* C *such that* $\int p_t(x)\phi(x)\,dx < C$ *for all* $t < t_0$.

Let $A = \mathrm{lin}\,\{p_t : t > 0\}$. *Then* A *is a commutative* $*$*-subalgebra of* $L^1(G)$. *Let* $\mathbf{A}$ *denote its closure in the* L^1 *norm. Then of course,* $p^t \in \mathbf{A}$.

From now on we assume that the group G is of polynomial growth (e.g. a compact extension of a nilpotent group; cf. [5]).

Let $\mathrm{Sp}_p = \{\lambda \in \mathbf{C}: (\lambda - L)^{-1}$ is a bounded operator on $L^p(G)\}^c$.

THEOREM 3. $\mathbf{A}$ *is symmetric, i.e.* $\mathrm{Sp}_{\mathbf{A}} f*f^*$ *is a real nonnegative for all* f *in* $\mathbf{A}$, *hence* $\mathrm{Sp}_p L = \mathrm{Sp}_2 L$ *for all* $1 \leqslant p < \infty$.

THEOREM 4. *The Gelfand space of* $\mathbf{A}$ *is naturally homeomorphic with* $\mathrm{Sp}_2 L$.

THEOREM 5. *There is an integer* r *depending on the group only such that the functions* $F \in C_c^r(R)$ *operate on the hermitian functions* f *in* A *into* $\mathbf{A}$.

THEOREM 6. *The algebra* $\mathbf{A}$ *is regular and the set of functions* f *in* $\mathbf{A}$, *such that* $\mathrm{supp}\,\hat{f}$ *is compact, is dense in* $\mathbf{A}$. *Hence*

(i) *Every proper ideal of* $\mathbf{A}$ *is a annihilated by a nonzero homomorphism of* $\mathbf{A}$ *into* $\mathbf{C}$.

(ii) *None of the* p_t*'s and* P^t*'s,* $t > 0$, *is contained in a proper left* (*or right*) *ideal of* $L^1(G)$.

(iii) *If* $u(t, x)$ *is a solution of* (4) *or* (6) *such that* $u(0, x) = \phi(x)$ *and* $|u(t, \cdot)|_\infty \leqslant C$, $t > 0$, *then, if for a* $t_0 > 0$, $\lim_{x\to\infty} u(t_0, x) = a$, *then* $\lim_{x\to\infty} f*\phi(x) = a\int f$.

Now let G be a connected, simply-connected nilpotent Lie group and let $\{\delta_s\}_{s>0}$ be a one-parameter group of dialations of G. Suppose that $X_1, \cdots, X_k$ generate LG as a Lie algebra and $\delta_{s*}X_j = sX_j$. For important examples where such a situation occurs see [1], [2], [6], [9], [12]. Let $d(\delta_s x) = s^r dx$, and let $L = X_1^2 + \cdots + X_k^2$. We then have

$$\delta_{s*} L = s^2 L. \tag{7}$$

From (7) we can easily deduce the following formulae

$$p_t(x) = t^{-r/2} p_1(\delta_{t^{-1/2}}(x)) \quad \text{and} \quad P^t(x) = t^{-r} p^1(\delta_{t^{-1}}(x)).$$

Clearly enough, also the whole algebra $\mathbf{A}$ is stable under automorphisms δ_s^*, $s > 0$.

The connection of p_t and P^t to the homogeneous norm functions (cf. [6]) and singular integrals on G we hope to study in a subsequent paper.

Details and proofs will appear elsewhere.

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