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APPROXIMATION THEOREMS, C^∞ CONVEX EXHAUSTIONS AND MANIFOLDS OF POSITIVE CURVATURE[1]

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In this note, we announce several approximation theorems on Riemannian manifolds as well as some of their consequences. First recall the relevant definitions. Let M be a Riemannian manifold. A function f: $M \longrightarrow \mathbf{R}$ is called *convex* iff its restriction to each geodesic is a convex function of one variable. A function f on M is called *strictly convex* iff given any compact $K \subset M$, there exists an $\epsilon > 0$ such that for every geodesic $\tau(t)$ parameterized by arc-length and defined on $(-s,s)$ with $\tau(0) \in K$, $f(\tau(s)) + f(\tau(-s)) - 2f(\tau(0)) > \epsilon s^2$ for all $s \in (0, \epsilon)$. A function f is *subharmonic* iff it is everywhere a subsolution of the Dirichlet problem, i.e. if B is a sufficiently small geodesic ball and u is a harmonic function such that $u = f$ on the boundary of B, then $u \geqslant f$ everywhere in B. If f is C^2, f is subharmonic iff $\Delta f \geqslant 0$, where $\Delta =$ the Riemannian metric Laplacian. For a $C^2 f$, we define f to be *strictly subharmonic* iff $\Delta f > 0$. If f is merely continuous, we say f is strictly subharmonic iff at each $x \in M$, there exists a C^2 strictly subharmonic f_0 near x such that $f - f_0$ is subharmonic near x.

Suppose M is a complex manifold, not necessarily equipped with a Riemannian metric. A real-valued C^2 function f on M is called *strictly plurisubharmonic* iff $d'd''f$ is a positive definite Hermitian form at each point. For a continuous function f, we say f is *strictly plurisubharmonic* iff at each point $x \in M$, one can find a C^2 strictly plurisubharmonic function f_0 near x such that $f - f_0$ is plurisubharmonic near x.

In the following, S will denote any one of the following subsets of the ring of real-valued functions on M: (A) convex functions, (B) continuous subharmonic functions, (C) continuous plurisubharmonic functions on a complex manifold, (D) continuous plurisubharmonic functions on a complex

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manifold which admits a continuous strictly plurisubharmonic function. S^∞ will then denote the C^∞ members of S. We also use S_+ to denote the subset of S consisting of strictly convex, or strictly subharmonic, or strictly plurisubharmonic functions, as the case may be. Naturally, we define: $S_+^\infty \equiv S^\infty \cap S_+$.

THEOREM 1. *S_+^∞ is dense in S_+ in the C^0 fine topology. Moreover, if $f \in S_+$ is Lipschitzian, then the approximating functions in S_+^∞ can all be chosen to be Lipschitzian with the same Lipschitz constant.*

The first statement of the theorem in the case $S = (C)$ is a previously known result given in [9]; the authors are indebted to Y.-T. Siu for bringing this paper to their attention.

A function $f: M \longrightarrow \mathbf{R}$ is called an *exhaustion function* iff it is continuous and each sublevel set $M_c \equiv \{x \in M: f(x) \leqslant c\}$ is compact. If f is an exhaustion function, then any continuous function that uniformly approximates f on M is itself an exhaustion function. Thus:

COROLLARY 1.1. *If M admits an exhaustion function belonging to S_+, then it also admits an exhaustion function belonging to S_+^∞.*

Basing our arguments on earlier works of Gromoll and Meyer [7] and Cheeger and Gromoll [1], we have observed in [6] that a complete, noncompact Riemannian manifold of positive curvature admits a strictly convex exhaustion function. Hence:

COROLLARY 1.2. *A complete noncompact Riemannian manifold of positive curvature admits a C^∞ strictly convex exhaustion function.*

By using Corollary 1.2, certain results in Gromoll and Meyer [7] can be generalized and, at the same time, given much simpler proofs. For example:

THEOREM 2. *Let M be a Riemannian manifold which admits a strictly convex (continuous) exhaustion function. Then M is diffeomorphic to euclidean space and the exponential map at each point of M is a proper map.*

The next theorem also follows easily from Theorem 1.

THEOREM 3. *Let M be a complete noncompact Riemannian manifold whose curvature is positive outside a compact set. Then*: (i) *M is isotopic to the interior of a compact manifold with boundary.* (ii) *If M is a Kähler manifold, then M is obtained from a Stein space by blowing up a finite number of* *points.* (iii) *If* M *is a Kähler manifold and if, in addition, the curvature is assumed to be everywhere nonnegative, then* M *is a Stein manifold.*

Part (ii) has been announced in [5] without indication of the method of proof. (iii) is an improvement of the main theorem of the same paper [5].

Returning now to the general approximation question for S itself, we have:

THEOREM 4. *Let* S *be* (B) *or* (D) *of the previously defined subsets of continuous functions. Then* S^∞ *is dense in* S *in the compact open topology. Moreover, if* $f \in S$ *is Lipschitzian, then an approximating sequence to* f *in* S^∞ *may be chosen to be Lipschitzian with uniformly bounded Lipschitz constants. Let* $S = $ (C), *let* $\epsilon\colon M \longrightarrow R^+$ *be any positive continuous function and* S_ϵ^∞ *be the set of* C^∞ *functions* $f\colon M \longrightarrow R$ *such that the eigenvalues (relative to a Hermitian metric) of the Levi form of* f *are* $> -\epsilon$ *at each point. Then* $\mathrm{Cl}\, S_\epsilon^\infty \supset S$ *(closure in the* C^0 *fine topology). Similarly if* $S = $ (A) *(or* (B)) *and* $S_\epsilon^\infty = $ *the set of* C^∞ *functions having the eigenvalues of their second covariant differential* $> -\epsilon$ *(respectively, having Laplacian* $> -\epsilon$) *then* $\mathrm{Cl}\, S_\epsilon^\infty \supset S$.

The proofs of Theorems 1 and 4 are based on a general procedure for passing from approximations near compact sets (cf. [4]) to global approximations (cf. [2]).

Combining Theorem 4 with the technique in [6], we can prove

THEOREM 5. *Let* M *be a Riemannian manifold which admits an exhaustion function which is, outside a compact set, Lipschitzian and subharmonic. Then* M *has infinite volume.*

COROLLARY 5.1. *Let* M *be either:* (i) *complete, noncompact, with positive curvature or* (ii) *complete simply connected, with nonpositive curvature everywhere. Then every properly immersed minimal submanifold of* M *has infinite volume. If in* (i), *one only assumes nonnegative curvature outside a compact set, then a noncompact properly immersed minimal submanifold again has infinite volume.*

The first part of this corollary generalizes the well-known fact that such minimal submanifolds are noncompact (folklore and Greene and Wu [3, I]).

Finally, we observe that Theorem 1 leads to a particularly simple proof of the Gauss-Bonnet theorem for complete noncompact Riemannian manifolds of dimension $\leqslant 4$, whose curvature is positive outside a compact set. Compared with the known result of Poor [8] and Walter [10], this result is at once more restrictive (we require positivity rather than nonnegativity of the curvature) and more general (the curvature in our theorem may be negative in a compact set). However, if we are willing to forego simplicity in the proof, then we can actually prove the following general theorem.

THEOREM 6. *Let M be a complete noncompact Riemannian manifold of dimension $\leqslant 4$ whose curvature is nonnegative outside a compact set. Then its total curvature does not exceed its Euler characteristic.*

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