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DIFFERENTIAL INEQUALITIES AND CARATHÉODORY FUNCTIONS

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ABSTRACT. The author proves a very general result from which it is possible to show that a regular function satisfying a differential inequality of a certain type is necessarily a Carathéodory function. This result has applications in the theory of univalent functions.

Let $\mathscr{P}$ denote the class of Carathéodory functions; that is, functions $p(z)=1+p_1z+p_2z^2+\cdots$ regular in the unit disc Δ, and for which $\operatorname{Re} p(z)>0$.

In a recent paper [2] it was shown that if $p(z)=1+p_1z+p_2z^2+\cdots$ is regular in Δ, with $p(z)\neq 0$ in Δ, and if α is a real number, then for $z\in\Delta$

$$\operatorname{Re}[p(z)+\alpha(zp'(z)/p(z))]>0 \Rightarrow \operatorname{Re} p(z)>0; \tag{1}$$

that is, $p(z)\in\mathscr{P}$.

In this note we replace the differential inequality in (1) by a much more general condition which will still imply that $p(z)$ is a Carathéodory function.

DEFINITION 1. Let $u=u_1+u_2i$ and $v=v_1+v_2i$, and let Ψ be the set of functions $\psi(u,v)$ satisfying:

(a) $\psi(u,v)$ is continuous in a domain D of $C\times C$;

(b) $(1,0)\in D$ and $\operatorname{Re}\psi(1,0)>0$;

(c) $\operatorname{Re}\psi(u_2i,v_1)\leqq 0$ when $(u_2i,v_1)\in D$ and $v_1\leqq -1/2(1+u_2^2)$.

We denote by Φ the subset of Ψ which satisfies (a), (b) and the following condition:

(c′) $\operatorname{Re}\psi(u_2i,v_1)\leqq 0$ when $(u_2i,v_1)\in D$ and $v_1\leqq 0$.

EXAMPLES. It is easy to check that each of the following functions are in Ψ.

$\psi_1(u,v)=u+\alpha v/u$, α real, with $D=[C-\{0\}]\times C$.

$\psi_2(u,v)=u^2+v$ with $D=C\times C$.

$\psi_3(u,v)=u+\alpha v$, $\alpha\geqq 0$, with $D=C\times C$.

$\psi_4(u,v)=u-v/u^2$ with $D=[C-\{0\}]\times C$.

$\psi_5(u,v)=-\ln(\frac{1}{2}-v)$ with $D=C\times\{(v_1,v_2)|v_1<\frac{1}{2}\}$.

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Note that ψ_1, ψ_2, ψ_3 and ψ_4 are also in Φ but $\psi_5 \notin \Phi$. The set Φ is thus a proper subset of Ψ. Though some generality is lost in considering the class Φ as opposed to considering Ψ, the former is much easier to work with algebraically.

DEFINITION 2. Let $p(z)=1+p_1z+p_2z^2+\cdots$ be regular in Δ and let $\psi \in \Psi$ with corresponding domain D. We denote by $\mathscr{P}(\psi)$ those functions $p(z)$ that satisfy:

(i) $(p(z), zp'(z)) \in D$, and

(ii) $\operatorname{Re} \psi(p(z), zp'(z))>0$,

when $z \in \Delta$.

Note that $\mathscr{P}(\psi)$ is not empty, since for all $\psi \in \Psi$ it is true that $p(z)=1+p_1z \in \mathscr{P}(\psi)$ for p_1 sufficiently small (depending on ψ). It appears further that most $\psi \in \Psi$ provide a large number of other functions in $\mathscr{P}(\psi)$.

Our main result is the following theorem.

THEOREM 1. *For any* $\psi \in \Psi$, $\mathscr{P}(\psi) \subset \mathscr{P}$.

In other words the Theorem states that if $\psi \in \Psi$, with corresponding domain D, and if $(p, zp') \in D$ then

$$\operatorname{Re} \psi(p(z), zp'(z)) > 0 \Rightarrow \operatorname{Re} p(z) > 0. \tag{2}$$

Since $\Phi \subset \Psi$, we immediately have the following Corollary.

COROLLARY. *For any* $\psi \in \Phi$, $\mathscr{P}(\psi) \subset \mathscr{P}$.

The proof of the Theorem is involved and will not be presented here. However an independent proof of the Corollary follows.

Let $p(z) \in \mathscr{P}(\psi)$, and assume there exists a point $z_0=r_0 \exp(i\theta_0) \in \Delta$ such that $\operatorname{Re} p(z) \geqq 0$ for $|z| \leqq r_0$, and $\operatorname{Re} p(z_0)=0$. Thus $p(z_0)=ai$, where a is a real number. We now show that $z_0p'(z_0)=k$, where $k \leqq 0$. Since the result is true if $p'(z_0)=0$, we need only consider the case $p'(z_0) \neq 0$. The curve $p(r_0e^{i\theta})$ is tangent to the imaginary axis at z_0, and so we have $\arg z_0p'(z_0)=\pi$; that is $z_0p'(z_0)=k$, where $k<0$. Hence at z_0 we have $\operatorname{Re} \psi(p, zp')=\operatorname{Re} \psi(ai, k)$ with a real and $k \leqq 0$. But this implies that $\operatorname{Re} \psi(p, zp') \leqq 0$ at $z=z_0$, which is a contradiction of the fact that $p(z) \in \mathscr{P}(\psi)$. Hence $\operatorname{Re} p(z)>0$ for $z \in \Delta$.

REMARKS. If we apply the Theorem (or the Corollary) to the example $\psi_1(u, v)$, we obtain condition (1). Applying it to ψ_2, ψ_3 and ψ_4 we obtain respectively:

$$\operatorname{Re}[p^2(z) + zp'(z)] > 0 \Rightarrow \operatorname{Re} p(z) > 0; \tag{3}$$

$$\operatorname{Re}[p(z) + \alpha zp'(z)] > 0, \quad \text{with } \alpha \geqq 0 \Rightarrow \operatorname{Re} p(z) > 0, \tag{4}$$

and

$$p(z) \neq 0 \quad \text{and} \quad \operatorname{Re}[p(z) - zp'(z)/p^2(z)] > 0 \Rightarrow \operatorname{Re} p(z) > 0. \tag{5}$$

We see that for different $\psi \in \Psi$ we can obtain different differential conditions for $p(z)$ to be a Carathéodory function. By appropriately choosing $\psi \in \Psi$ we can define many new subclasses of $\mathscr{P}$ and can prove many properties of the class $\mathscr{P}$.

The theorem has many applications in the theory of univalent functions. If we set $p(z)=zf'(z)/f(z)$ in Theorem 1, we see from (2) that each $\psi \in \Psi$ generates a subclass of starlike functions. In particular $\psi_1(u, v)=u+\alpha v/u$ generates the class of alpha-convex functions [**2**]. Similarly by setting $p(z)=e^{i\gamma}zf'(z)/f(z)$, where $|\gamma|<\frac{1}{2}$, or $p(z)=f'(z)/g'(z)$, where $g(z)$ is convex, and using slightly modified forms of Definitions 1 and 2 and Theorem 1, we can generate many new subclasses of spiral-like and close-to-convex functions, respectively. These results, the proof of Theorem 1, and other applications will appear in a forthcoming paper [**1**].

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