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BOOK REVIEW

Classical Banach spaces, by Joram Lindenstrauss and Lior Tzafriri, Springer-Verlag, Lecture Notes in Math., vol. 338, Berlin, 1973. ix+243 pp.

Banach space theory has seen considerable change in recent years and these excellent lecture notes provide a highly readable and up-to-date survey of this area. The authors' objectives are stated in the preface: "The main purpose of these lecture notes is to give an outline of that part of Banach space theory which deals with properties of special and important classes of spaces. We have tried to present the main methods used in the theory, as well as the principal ideas involved in the proofs of the basic results. The lecture notes contain some well known and classical theorems but the main subject matter consists of recent results and research directions. Many open problems are mentioned throughout these notes."

Though mainly directed at Banach space enthusiasts, these lecture notes should appeal to a far wider audience. In particular, much of the recent progress on L_p-spaces is closely connected to various topics in probability theory (infinitely divisible laws, stable laws, martingale inequalities) and harmonic analysis ($\Lambda(p)$-sets, multiplier theory), while the indices of an Orlicz space (introduced by Boyd in connection with interpolation theory) are now known to be intimately related to the subspace structure of these spaces. Techniques are also borrowed from topological dynamics, integral geometry, nonstandard analysis, and other areas. With this in mind, and in view of the fact that an excellent, technical review of these lecture notes has recently been given by G. Köthe [Zentralblatt für Math. **259** (1974), 282–284], I shall aim this review at the nonspecialist. Accordingly, I shall ignore many important, but "hard-core" theorems, and emphasize those that can most easily be described in everyday terms. I hope that in so doing I shall still convey to the reader the spirit and flavor of these important lecture notes.

The notes are divided into two main parts: I. Sequence spaces, II. Function spaces. Both parts begin with an introductory chapter of a rather general nature. The main aim of these introductory sections "is to present the definitions and illustrate some basic notions which are usually not discussed in a standard first course in functional analysis $\cdots$. Because of their nature, these chapters are brief and do not contain many details."

Despite the authors' disclaimer, the first of these chapters gives what is perhaps the most readable introduction in print to Schauder bases, while the second provides a rather useful survey of Banach lattices.

The remaining sections, which contain the main subject matter of the notes, are treated in much greater detail. Even then, the proofs of several theorems are too long to have been included in their entirety, and often, in such cases, the authors have striven to give the reader somewhat more than a mere inkling of the ideas involved. This is an extremely valuable feature of these notes.

Using the introductory section on bases, the authors turn their attention to a detailed study of the spaces l_p and c_0. Numerous results are given which single out these spaces from more general classes of Banach spaces. Perhaps the highlight of this section is Enflo's solution to the famous approximation problem, the elegant presentation of Davie being given here in some detail. It is interesting to note that the counterexample can be taken to be a subspace of c_0, or of l_p for $p>2$ (the situation for $p<2$ is still open). This section also contains the remarkable inequality of Grothendieck. It is to be hoped that these lecture notes will succeed in broadcasting this fundamental result, which has not yet received the attention it deserves from analysts at large (though it has found recent application to abstract harmonic analysis in, for example, the work of J. E. Gilbert).

The next chapter deals with a general class of Banach spaces, those having a symmetric basis. This class contains c_0 and l_p, $1\leqq p<\infty$, (as well as the Lorentz and Orlicz spaces) and the results presented here help place the above theory in a more general setting. Thus, roughly speaking, it is shown that the only Banach spaces with a unique unconditional basis are l_1, l_2 and c_0, while all the spaces c_0 and l_p have a unique symmetric basis.

Part I closes with an account of Orlicz sequence spaces. This section, based heavily on the authors' own work, is of a more technical nature, but well worth reading for the number and variety of techniques involved. Aside from any intrinsic interest they may possess, Orlicz sequence spaces have been extremely useful in showing how the more familiar spaces, l_p and c_0, fit into the general theory of Banach spaces. In particular, the conjecture that every Banach space contains a complemented copy of c_0 or (some) l_p was shown to be false in this setting. On the other hand, via a clever application of the Schauder-Tychonoff fixed point theorem, the authors show that every Orlicz sequence space contains a copy of c_0 or (some) l_p. This lent much weight to the conjecture that every Banach space has this property (a result, unfortunately, now known to be false).

After the brief introductory survey of Banach lattices, Part II continues with a discussion of various characterizations of the classical spaces among

Banach lattices. Comprehensive results on both the isometric and isomorphic theories are treated in some detail. The chapter closes with a discussion of ultraproducts of Banach spaces which should be of interest to students of nonstandard analysis. A compactness argument is used, both here and in succeeding chapters, to obtain global information on a space from the properties of its finite dimensional subspaces.

Most of the next chapter deals with structural properties of subspaces if $L_p(0, 1)$. In particular, the problem of determining when $L_p(0, 1)$ embeds in $L_r(0, 1)$ is studied in great detail. Partial results along these lines were known to Banach, but the general problem was solved only recently. The two deepest results are: There is no (isomorphic) embedding if $r>p>2$; there is an (isometric) embedding if $1\leqq r<p<2$. The negative result, due originally to Paley, is presented here using techniques of Kadeć and Pełczyński. The positive one, proved in various degrees of generality by several authors, uses the theory of negative definite functions. An alternative approach, using p-stable probability distributions, has been given by Rosenthal. This chapter also contains a brief discussion of the isometric theory, some recent results on uniformly convex spaces, and applications of the Riesz-Thorin and Marcinkiewicz interpolation theorems to the study of Schauder bases in $L_p(0, 1)$.

The next chapter is devoted to a study of the continuous function spaces, $C(K)$. This theory is radically different from that of the $L_p(\mu)$-spaces, not only in the methods used but also in the questions asked. For example, the subspace structure of $L_p(0, 1)$, $1\leqq p<\infty$, has been studied extensively and is reasonably well understood, whereas *every* separable Banach space occurs (even isometrically) as a subspace of $C(0, 1)$. (The complemented subspaces, however, form an interesting class and there is some discussion related to the problem: Is every complemented subspace of $C(K)$ isomorphic to $C(H)$ for some H?) The first half of the chapter is devoted to the isometric theory, the second to the isomorphic theory. Again, the two theories are basically different. On the one hand there is the famous Banach-Stone theorem: $C(K)$ and $C(H)$ are isometric if and only if K and H are homeomorphic, while Milutin has shown: K metric, uncountable implies that $C(K)$ is isomorphic to $C(0, 1)$. Attempts to isolate the $C(K)$-spaces have led Lindenstrauss to a comprehensive study of the extension properties of compact operators. In these results, and in many others, a new and important class of spaces emerges, the so-called L_1-preduals (those X for which X^* is isometric to $L_1(\mu)$ for some measure μ). This class, which is strictly larger (even in the isomorphic sense) than the class of $C(K)$-spaces, seems to provide the most natural framework in which to study the isometric properties of the $C(K)$-spaces. The L_1-preduals that are isometric to some $C(K)$ have been completely described;

the corresponding isomorphism problem is much more difficult and remains largely unsolved. The chapter closes with a detailed discussion of injective Banach spaces, a study which leads inevitably to interesting and difficult questions concerning nonseparable spaces.

The notes close with a chapter on $\mathscr{L}_p$-spaces. The precise definition of these spaces is too lengthy to be stated here—roughly speaking, these are the Banach spaces whose finite-dimensional subspaces are close to the finite-dimensional subspaces of l_p-spaces. They include $L_p(\mu)$ and $C(K)$ as special cases, and play a vital role in the isomorphic classification of these spaces and their subspaces. For example, every complemented subspace of an $L_p(\mu)$-space, $1<p<\infty$, turns out to be l_2 or an $\mathscr{L}_p$-space, and, conversely, every $\mathscr{L}_p$-space can be realized as a complemented subspace of some $L_p(\mu)$-space. Their global properties are quite complicated and many natural questions remain unsolved. However, much is known, and we cite the remarkable result of W. B. Johnson, H. P. Rosenthal and M. Zippin: Every separable $\mathscr{L}_p$-space has a basis. This is essentially the only known, general result asserting the existence of a basis for a class of Banach spaces not defined in an explicit fashion. A somewhat perplexing feature of the area is the apparent lack of examples. Rosenthal has constructed several (finitely many) different isomorphism types of $\mathscr{L}_p$-spaces using 3-valued independent random variables, but much remains to be done in this direction. Nevertheless, the presently known results have more than justified the introduction of the $\mathscr{L}_p$-spaces, and perhaps the crowning achievement of this theory is the authors' solution to the famous complemented subspaces problem: A Banach space, each of whose (closed) subspaces is complemented, must be isomorphic to Hilbert space.

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