WEIL REPRESENTATIONS AND CUSP FORMS ON UNITARY GROUPS

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Introduction. In this note we announce some results on the character theory of the finite unitary groups. The main result is the decomposition of the restriction of a certain class function J to a maximal parabolic subgroup as a tensor product. This result is analogous with those of [4] for the general linear group $GL(n, q)$, with the difference that the components of the tensor product are more complicated here. In fact, one has a Levi decomposition $P=MH$ of the maximal parabolic subgroup P, where M is a unitary group of lower dimension, and H is isomorphic to the "unitary Heisenberg group" (see below); one therefore (see [3], [6], [7]) obtains "Weil representations" of P, and these are instrumental in the decomposition of J.

Applications include an inductive proof that the restriction of J to the unipotent subgroup of the unitary group is a proper character, and that J has a nontrivial projection to the space of cusp forms on the unitary group in odd dimensions.

In some forthcoming work of Lusztig, the main theorem is used to show that J is in fact a proper irreducible discrete series character of the unitary group, verifying a long-standing conjecture of Ennola [1].[1]

1. **The unitary group.** We consider the finite unitary groups realized as follows: Let G be $GL(n, q^2)$, and consider the automorphism σ of G given by

$$\sigma = {}^{-} \circ -1 \circ t \circ Y,$$

where $\circ$ denotes composition, $^{-}$ is conjugation (taking the matrix elements to the qth power), t is transposing of matrices and Y is conjugation by the matrix

$$\begin{bmatrix} & & & & 1 \\ & 0 & & 1 & \\ & & \cdot & & \\ & & \cdot & & \\ & \cdot & & 0 & \\ 1 & & & & \end{bmatrix}.$$

[1] *Added in proof.* The author has recently learned that Ennola's conjecture for J_n is still open.

Then denote by $U(n, q^2)$, or U_n when unambiguous, the group $\{g \in G | g^\sigma = g\}$.

The subgroup of U_n with which we are concerned is in fact a subgroup of small index in a maximal parabolic subgroup. We do not take the whole parabolic subgroup to reduce the volume of unnecessary detail. The group P we consider is that consisting of matrices g in U_n of the shape in Figure 1.

$$g = \begin{bmatrix} 1 & \beta & x \\ 0 & t & \alpha \\ 0 & 0 & 1 \end{bmatrix},$$

FIGURE 1

where $t \in U_{n-2}$, α and $\beta \in V(n-2, q^2)$ (a vector space of dimension $n-2$ over $GF(q^2)$) and $x \in GF(q^2)$.

The condition $g^\sigma = g$ shows that β is determined by α, and x satisfies $x + \bar{x} = -\langle \alpha, \alpha \rangle$, where $\langle \ , \ \rangle$ denotes the hermitian form on $V(n-2, q^2)$ given by

$$\langle \alpha, \alpha' \rangle = \sum_{i=1}^{n-2} \alpha_i \bar{\alpha}'_{n-1-i}.$$

We therefore denote g above by (t, α, x).

The subgroup of P, consisting of $\{(t, \alpha, x) | \alpha = 0, x = 0\}$, is clearly isomorphic to U_{n-2}, and its normal complement $H_n = \{(t, \alpha, x) | t = I\}$ is called the unitary Heisenberg group. We have $P = U_{n-2} H_n$, the Levi decomposition.

2. **The class function** J. The Jordan canonical form shows that a conjugacy class in $GL(n, q^2)$ is given by a partition-valued function on the set of irreducible polynomials over $GF(q^2)$. Wall [5] has shown that two elements of $U(n, q^2)$ are conjugate if and only if they are conjugate in $GL(n, q^2)$, and from this it is not hard to deduce that the conjugacy classes of U_n are parametrized by partition-valued functions on the set of U-irreducible polynomials over $GF(q^2)$ (see [1]), where a U-irreducible polynomial is either irreducible or a product of two related irreducible polynomials.

The work of Green [2] and subsequent authors showed that there is a family of distinguished characters of the general linear groups, called discrete series, which are in some sense necessary and sufficient for building the character table. In 1962, Ennola, by "twisting" the functions in Green's work, produced an orthonormal basis for the set of class functions on $U(n, q^2)$, which he conjectured were the irreducible characters of U_n. It is with the twisted form of a discrete series character of $GL(n, q)$ that we are concerned. We define it simply by its values on the classes.

Let ψ be a fixed character of the multiplicative group $GF(q^{2n!})^*$. Then define the class function $J=J_n$ by $J_n(c)=0$ unless $c=(f^\nu)$ (f being a U-irreducible polynomial), and

$$J_n(f^\nu) = k(\nu, (-q)^{d(f)})S_{d(f)}(\gamma),$$

where: $p(\nu)$ is the number of parts of the partition ν,

$$k(\nu, t) = (1 - t)(1 - t^2) \cdots (1 - t^{p(\nu)-1}),$$

$d(f)=$degree of the polynomial f, γ is a root of f, and

$$S_{d(f)}(\gamma) = \psi(\gamma) + \psi(\gamma^{-q}) + \psi(\gamma^{q^2}) + \cdots + \psi(\gamma^{(-q)^{d(f)-1}}).$$

3. **The Weil representations.** The centre Z of H_n is $\{(t, \alpha, x)|t=I, \alpha=0\}$ and is isomorphic to the additive group $GF(q)^+$; Z is also the derived group of H_n. Hence H_n has $|H_n/Z|=q^{2n-4}$ distinct irreducible characters of degree one. It is easily seen further than H_n has precisely $q-1$ other irreducible representations which are parametrized by the nontrivial characters χ of $GF(q)^+$. We denote these by ρ_χ, and their characters by Q_χ. Now the group U_{n-2} centralises the representation ρ_χ, and we therefore obtain a projective representation $\tilde{\rho}_\chi$ (called a Weil representation) of $P=U_{n-2}H_n$, whose restriction to H_n is ρ_χ. Moreover we can prove

PROPOSITION 3.1.   (i) *There is a representation $\tilde{\rho}_\chi$ of P whose restriction to H_n is ρ_χ, and whose character W_χ takes rational values on unipotent elements.*

(ii) *The rationality condition in* (i) *determines W_χ uniquely on all unipotent subgroups of P.*

The proof involves a characterisation of $\tilde{\rho}_\chi$ as an induced representation on unipotent subgroups, and the rationality comes from a consideration of the set of all W_χ as a whole.

In the determination of the values of W_χ, the following lemma is of central importance (cf. Howe [3]).

LEMMA 3.2.   *Suppose $V=V_1 \perp V_2$ (where $V=V(n-2, q^2)$ and $\perp$ denotes the orthogonal direct sum with respect to $\langle \ , \ \rangle$), and let T be a unipotent group of unitary operators on V such that $TV_i=V_i$ ($i=1, 2$). Then if $H_i=\{(1, \alpha, x)|\alpha \in V_i\}$, and W_i^* denotes the lift of the (unique) Weil character of TH_i to TH, where W_i corresponds to χ, we have*

$$W_\chi = W_1^* \otimes W_2^*.$$

4. **The main theorem.** Lemma 3.2 can be applied in various situations to give complete information about the values of W_χ, which furnishes a proof of

THEOREM A. *Let J_n be the function defined in §2 above, and let J_{n-2} be the corresponding function (for the same ψ) on U_{n-2}. Then we have (for appropriate choice of W_χ)*

$$J_n = J_{n-2}^P - J_{n-2}^* + (-1)^{n-1} \sum_\chi J_{n-2}^* \otimes W_\chi$$
$$= J_{n-2}^* \otimes \theta,$$

where J_{n-2}^P denotes the induced function, J_{n-2}^* is the lift (or pullback) of J_{n-2} from U_{n-2} to P, and θ is the appropriate character of P (given by the first line of the formula).

5. Applications.

COROLLARY 1. *J_n is a proper character (i.e. the character of a representation) of the maximal unipotent subgroup S of U_n.*

The proof is by induction on n and depends on the observation that a copy of S is contained in P, as the group of upper unitriangular matrices in U_n. This is because the automorphism σ preserves the group of upper unitriangular matrices of $GL(n, q^2)$. Theorem A applies equally well to S, where W is interpreted as its restriction to S and so automatically gives a proof by induction, since θ is the character of a representation of S.

COROLLARY 2. *For n odd, J_n has a nontrivial projection to the space of cusp forms on U_n.*

The proof of this involves showing that certain scalar products are zero— viz. if R is the radical of a maximal parabolic subgroup of U_n, it is required that $(J_n, I_R^S)=0$. Theorem A, together with some general techniques of character theory, reduces the proof of this to the corresponding result for J_{n-2}.

As mentioned in the introduction,[2] the work of Lusztig on the Brauer lifting of the natural representation of U_n reduces the problem of showing that J_n is a proper character of U_n to Corollary 2 above, showing that here, J_n is indeed a discrete series character. The generation of the whole character table of U_n from the family of J_n is complicated in this case, however, because the polynomials appearing in Ennola's table do not correspond to procedures such as inducing characters (i.e. are not structure constants in a Hall algebra) as is the case in $GL(n, q)$.

Further details of this work shall appear in a forthcoming paper of the author.

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[2] See footnote 1.

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