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A UNIFIED APPROACH TO GENERALIZED INVERSES OF LINEAR OPERATORS: II. EXTREMAL AND PROXIMAL PROPERTIES

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1. **Introduction.** This note is a sequel to [**5**]; the notations are the same. An important property of the Moore-Penrose inverse $T^\dagger$ of a matrix, or a bounded linear operator, or a densely-defined closed linear operator T on a Hilbert space is the relation of $T^\dagger$ to extremal solutions of the equation $Tu=y$ (see [**2**], [**4**], [**7**]). We develop proximal properties of generalized inverses in normed spaces within the general setting of [**5**] and demonstrate their relations to extremal, minimal, and best approximate solutions.

2. **Preliminaries.** Let N be a normed linear space. Let sp A denote the span of a set A and $d(x, A)$ the distance from x to $A \subset N$. The following definition of *orthogonality* is used: $x \perp y$ means $d(x, \mathrm{sp}\{y\}) = \|x\|$, and $B \perp A$ means $d(x, A) = \|x\|$ for each $x \in B$. Note that this orthogonality is not a symmetric relation. Let M be a subspace of N which has a topological complement, and consider the affine manifold $\mathscr{P}_M = \{P \in \mathscr{L}(N): P^2 = P \text{ and } \mathscr{R}(P) = M\}$.

PROPOSITION 1. *Let $P_0 \in \mathscr{P}_M$. The following statements are equivalent*:
(a) *P_0 is the nearest point in $\mathscr{P}_M$ to I, and $\|I - P_0\| = 1$.*
(b) $\mathscr{N}(P_0) \perp \mathscr{R}(P_0)$.
(c) *For each $x \in N$, $P_0 x$ is the nearest point in M to x.*

If there exists a $P_0 \in \mathscr{P}_M$ such that any (and hence all) of the statements in Proposition 1 hold, we say that M is an *orthogonally-complemented* subspace of N. We emphasize that if $M = \mathscr{R}(P_0)$ is orthogonally complemented by $\mathscr{N}(P_0)$, it is not necessarily true that $\mathscr{N}(P_0)$ is orthogonally complemented by $\mathscr{R}(P_0)$. Also, orthogonal complements are not necessarily unique. Hilbert spaces are an exceptional case; if M is a closed

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subspace of a Hilbert space, then $M^\perp$ is the unique orthogonal complement of M in the above sense, and also $M^\perp$ has M as an orthogonal complement. In this case the corresponding projectors will be selfadjoint. Finally we remark that in a normed linear space every one-dimensional subspace is an orthogonal complement for some hyperplane, for let $S=\text{sp}\{e\}$, $\|e\|=1$. Take (by the Hahn–Banach theorem) $f\in N^*$ with $\|f\|=1$ and $f(e)=1$. Define P by $Px=x-f(x)e$. Then $\mathscr{R}(P)=\mathscr{N}(f)$ and $\|I-P\|=1$. In general, in normed linear spaces, orthogonally-complemented subspaces are rare.

3. **Main results.** Let $\Lambda(V, W)$ denote the space of all linear maps from a vector space V into a vector space W. If $T\in\Lambda(V, N_2)$ where N_2 is a normed space, then $v_0\in V$ is called an *extremal solution* of the equation $Tv=y$ if $v=v_0$ minimizes $\|Tv-y\|$. If $V=N_1$ is also a normed space, then an extremal solution of minimal norm is called a *best approximate solution* or a *least extremal solution*. The equation $Tv=y$ need not have an extremal solution for each y, and the existence of an extremal solution does not imply the existence of a best approximate solution.

DEFINITION 2. Let N_2 be a normed linear space, and let $T\in\Lambda(V, N_2)$. If $U=T^\dagger_{r,Q}$ is a R-T.P.I. of T where $\|I-Q\|=1$, we call U a *right-orthogonal partial inverse* (abbreviated as R-O.P.I.) of T. On the other hand, if $\mathscr{D}(T)=V$ is contained in a normed linear space N_1, $\mathscr{R}(T)$ is in a (algebraic) vector space W, and $U=T^\dagger_{l,P}$ is a L-T.P.I. of T, where $\|I-P\|=1$, we call U a *left-orthogonal partial inverse* (L-O.P.I.). If U is both a L-O.P.I. and a R-O.P.I. we call U an *orthogonal partial inverse* (O.P.I.) of T.

Note that by Proposition 1, the existence of a R-O.P.I. implies that $\overline{\mathscr{R}(T)}$ is an orthogonally-complemented subspace of N_2. Also, the existence of a L-O.P.I. implies that $\mathscr{C}_P(T)\perp\mathscr{N}(T)$ since $\mathscr{N}(P)\perp\mathscr{R}(P)$ by Proposition 1 and since $\mathscr{N}(T)\subset\mathscr{R}(P)$ and $\mathscr{C}_P(T)\subset\mathscr{N}(P)$ by [**5**, Theorem 2.4, Definition 2.2]. Thus, in general, the existence of an O.P.I. is a rare phenomenon in normed spaces without inner products.

If $N_2=H_2$ is a Hilbert space, there is a unique Q with range $\overline{\mathscr{R}(T)}$ satisfying $\|I-Q\|=1$. Thus for $T\in\Lambda(V, H_2)$, a R-O.P.I. always exists. If $N_1=H_1$ is also a Hilbert space, then T has an O.P.I. if and only if T is decomposable with respect to a selfadjoint projector P in the sense of [**5**, Definition 2.2].

The importance of right-orthogonal, left-orthogonal, and orthogonal partial inverses lies in their connection with the existence of extremal solutions, minimal solutions, and best approximate solutions of the equation

$$Tx = y. \tag{1}$$

THEOREM 3. *If U is a right-orthogonal partial inverse of $T \in \Lambda(V, N_2)$, then for $y \in \mathscr{D}(U)$ the equation* (1) *has $v = Uy$ as an extremal solution. If, in addition, nearest points from $\overline{\mathscr{R}(T)}$ are unique* (*in particular if N_2 is strictly convex*), *then the existence of an extremal solution implies that $y \in \mathscr{D}(U) = \mathscr{R}(T) \oplus \mathscr{N}(Q)$.*

THEOREM 4. *If U is a left-orthogonal partial inverse of T and the equation* (1) *has a solution, then Uy is a* (*not-necessarily unique*) *solution of minimal norm.*

THEOREM 5. *If U is an orthogonal partial inverse of T, and if in N_2 nearest points from $\overline{\mathscr{R}(T)}$ are unique, then for all $y \in \mathscr{D}(U)$, $x = UTUy = T^{\dagger}_{P,Q}y$ is a* (*not-necessarily unique*) *best approximate solution of* (1).

COROLLARY 6. *Let H_1 and H_2 be Hilbert spaces and let $T: \mathscr{D}(T) \subset H_1 \to H_2$. Then* (1) *has an extremal solution if and only if the orthogonal projection of y onto $\overline{\mathscr{R}(T)}$ lies in $\mathscr{R}(T)$. In this case* (1) *has a* (*unique*) *best approximate solution if and only if the orthogonal projection of any* (*every*) *extremal solution onto $\overline{\mathscr{N}(T)}$ lies in $\mathscr{N}(T)$. If T has an orthogonal partial inverse, then* (1) *has a unique best approximate solution whenever it has an extremal solution.*

This corollary basically contains the results given in Erdelyi and Ben-Israel [**2**, §2]. For matrices, it is well known that $\|Ax-b\|_2$ is minimized by $x=Ub$ if U satisfies $AUA=A$ and AU is selfadjoint (see e.g. Rao and Mitra [**8**]). This is a special case of Theorem 3. Also if $Ax=b$ has a solution, then the unique solution of smallest norm is $x=Ub$ where $AUA=A$ and UA is selfadjoint. This follows as a special case of Theorem 4 plus the use of strict convexity which gives uniqueness. Finally the well-known results for a bounded or densely-defined closed linear operator on a Hilbert space with closed or nonclosed range (see, e.g., [**4**]) are also corollaries.

In view of Theorem 5, which gives the existence of best approximate solutions, it is useful to consider another kind of generalized inverse.

DEFINITION 7. Let $T \in \Lambda(N_1, N_2)$, and consider a $y \in N_2$ such that $Tx=y$ has a best approximate solution in N_1. We define

$$T^{\partial}(y) = \{x \in N_1 \mid x \text{ is a best approximate solution to } Tx = y\}$$

and call the set-valued mapping $y \to T^{\partial}(y)$ the *metric generalized inverse*. Here $\mathscr{D}(T^{\partial}) = \{y \in N_2 | Tx=y \text{ has a best approximate solution in } N_1\}$. A (in general nonlinear) function $T^{\sigma}(y) \in T^{\partial}(y)$ is called a *selection* for the metric generalized inverse.

From Theorem 3 we see that if T has a right-orthogonal partial inverse and if nearest points from $\overline{\mathscr{R}(T)}$ are unique, then $\mathscr{D}(T^{\partial}) \subset \mathscr{R}(T) \oplus \mathscr{N}(Q)$.

If in addition T has an orthogonal partial inverse U, then $\mathscr{D}(T^\partial)=\mathscr{D}(U)=\mathscr{R}(T)\oplus\mathscr{N}(Q)$ and $T^\dagger_{P,Q}=UTU$ is a linear selection for T^∂.

4. **Remarks and special cases of the metric generalized inverse.** (i) Holmes [3] defines the generalized inverse of $T\in\mathscr{L}(N_1, N_2)$ to be T^∂ in the case where $T^\partial(y)$ is always a singleton and gives some conditions which imply that T^∂ is densely defined and/or continuous. He uses this approach to obtain the (linear) generalized inverse in the Hilbert space case.

(ii) Newman and Odell [6] have studied T^∂ in the finite-dimensional case where the norms are strictly convex. In this case, there is always a unique best approximate solution of $Lx=y$ given by $B=(I-F)ME$ where M is any partial inverse of L, E is the nearest point map onto $\mathscr{R}(A)$, and F is the nearest point map onto $\mathscr{N}(A)$. Some of the results of Erdelsky [1] are also subsumed by our theorems.

(iii) Erdelyi and Ben-Israel [2] considered the case of an arbitrary linear mapping with domain in a Hilbert space H_1 and range in a Hilbert space H_2. In this case $T^\partial(y)$ is always a singleton and T^∂ is a linear map. They used the name *g-inverse*. Here $\mathscr{D}(T^\partial)=T(\mathscr{C})\oplus\mathscr{R}(T)^\perp$ where $\mathscr{C}=\mathscr{D}(T)\cap\mathscr{N}(T)^\perp$, and $\mathscr{R}(T)\not\subset\mathscr{D}(T^\partial)$ in general. Let

$$\tilde{T} = T\,|\,\mathscr{D}_0 \quad \text{where } \mathscr{D}_0 = \mathscr{N}(T)\oplus\mathscr{C}.$$

Then $\tilde{T}$ has an orthogonal generalized inverse $\tilde{T}^\dagger$ in the sense of Definition 2 with domain $T(\mathscr{C})\oplus T(\mathscr{C})^\perp$, and the g-inverse turns out to be $T^\partial=\tilde{T}^\dagger|\mathscr{D}(T^\partial)$. If T is decomposable with respect to a selfadjoint projector, then $T^\partial=T^\dagger$; see Corollary 6.

(iv) The problem of obtaining selections with nice properties for the metric generalized inverse of linear and nonlinear operators merits study.

The proofs and some additional results will be published elsewhere.

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