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A LOWER ESTIMATE FOR EXPONENTIAL SUMS

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1. **Introduction.** In this note we present two theorems on exponential sums (see Theorems 1 and 2 below). Although seemingly unrelated, both results are motivated by the study of a certain type of lower estimates of exponential sums in the complex domain. Thus while Theorem 2 is related to the validity of this estimate for all *discrete* exponential sums[2], Theorem 1 essentially says that even a milder estimate of this kind does not hold for a whole class of *continuous* exponential sums (i.e. for certain Fourier transforms).

In addition to the usual notation of the theory of distributions (cf. [2], [3], [7]), the following symbols will be used throughout this note. Given a distribution $\Phi \in \mathscr{E}' = \mathscr{E}'(\boldsymbol{R}^n)$, the symbol $[\Phi]$ ($\{\Phi\}$ resp.) denotes the convex hull of the support of Φ (singular support of Φ, resp.). For $A \subset \boldsymbol{R}^n$, h_A is the supporting function of A, i.e. $h_A(\lambda) = \sup_{x \in A} \langle x, \lambda \rangle$, $\lambda \in \boldsymbol{R}^n$. For $\zeta \in \boldsymbol{C}^n$ and $r > 0$, $\Delta = \Delta(\zeta; r)$ is the closed polydisk with center ζ and radius r; and, if $g(\zeta')$ is any continuous function on $\Delta(\zeta; r)$, we shall write

$$|g(\zeta)|_r = \max_{\zeta' \in \Delta} |g(\zeta')|. \tag{1}$$

2. **Indicators of smooth convex bodies.**

DEFINITION. Let $\Phi \in \mathscr{E}'$ be such that

$$\{\Phi * \Psi\} = \{\Phi\} + \{\Psi\} \qquad (\forall \Psi \in \mathscr{E}'). \tag{2}$$

Then Φ will be called a *good convolutor*.

The relationship of being a good convolutor to the solvability of the convolution equation $\Phi * u = f$ in the appropriate distribution spaces was discovered by L. Hörmander [7], and since then it was discussed by several authors (for references, cf. [2, Chapter I]). However, it is usually not easy to decide whether a given distribution Φ is a good convolutor or not.

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[2] And more generally, for all exponential polynomials.

Moreover, few good convolutors are known, and as Theorem 1 below will indicate, even distributions of a very simple nature may fail to be good convolutors.

It can be shown [**4**, Proposition 2] that the following condition on $\hat{\Phi}$ is sufficient for Φ to be a good convolutor:

CONDITION (R_ω). There exist constants $t \geqq 0$, $r>0$, $c>0$ and A real (all depending on Φ) so that (cf. (1))

$$|\hat{\Phi}(\zeta)|_r \geqq c(1+|\xi|)^A \exp(h_{[\Phi]}(\eta)) \tag{3}$$

for all $\zeta=\xi+i\eta \in \boldsymbol{C}^n$ such that $|\xi| \geqq t$ and $|\eta|>t \log(1+|\xi|)$.

Since any distribution Φ with finite support satisfies condition (R_ω) (cf. [**4**, Proposition 6]), we thus obtain a result of Hörmander [**7**], [**8**], according to which all distributions with finite support are good convolutors. This in turn can be used to prove the following statement (cf. [**4**, Proposition 6]):

Let P be an arbitrary compact convex polyhedron in $\boldsymbol{R}^n$ and χ_P the distribution defined by the characteristic function of P. Then χ_P satisfies condition (R_ω), *hence χ_P is a good convolutor. The same conclusion holds for the surface measure $\chi_{\partial P}$ of density* 1, *i.e.*

$$\chi_{\partial P}(\phi)=\int_{\partial P} \phi(x)\, ds_x, \qquad (\phi \in \mathscr{E})$$

where ds_x is the surface element.

It seems natural to ask whether this proposition holds for smooth convex bodies P as well. At the first glance it seems that it does. Indeed, if, for instance, P is any ellipsoid in $\boldsymbol{R}^n$, then the distribution $\Phi=\chi_P$ satisfies the following weaker version of (2) (cf. the concluding remark in [**5**]):

$$\{\Psi\} \subseteq \{\Phi * \Psi\} - \{\Phi\} \qquad (\forall \Psi \in \mathscr{E}'). \tag{2*}$$

Therefore, it is rather surprising that this particular Φ is not a good convolutor [**5**, Proposition 4]. The following theorem sheds more light on this peculiar situation.

THEOREM 1. *Let P be a convex body in $\boldsymbol{R}^n$ ($n>1$) with a C^∞-boundary ∂P. Moreover, it is assumed that the Gaussian curvature of ∂P never vanishes, i.e. $K(x)>0$ for every $x \in \partial P$. Then neither χ_P nor $\chi_{\partial P}$ is a good convolutor.*

REMARK. Both assumptions on ∂P (i.e. smoothness and $K>0$) can be substantially relaxed.

The proof of Theorem 1 is based on a detailed study of the asymptotic behavior of the functions $\hat{\chi}_P$ and $\hat{\chi}_{\partial P}$ in the complex domain. For ζ real, estimates of this kind were previously derived by numerous authors (cf. [**9**], [**10**], [**11**] and the references given in [**10**], [**11**]). However, for our purposes these estimates must be sharpened. As an illustration, consider the case of the convex surface $S=\partial P$. Given $\zeta=\xi+i\eta\in C^n$ with $\xi\neq 0$, write $r=|\xi|$ and consider $\zeta=r\omega+i\eta$ with ω fixed. Let $x^j=(x_1^j,\cdots,x_n^j)\in S$ $(j=0,1)$ be the points

$$x_\nu^j=\partial h_S((-1)^j\xi)/\partial\xi_\nu \qquad (\nu=1,\cdots,n).$$

Fix arbitrarily the open subsets S^k $(k=0,1,2)$ of S so that $S=\bigcup_k S^k$, $S^0\cap S^1=\varnothing$, $x^j\in S^j\backslash S^2$ $(j=0,1)$. Then for any $q>n/2$ and $\nu>0$ there exist positive numbers a_j, b_j and c_ν such that

$$\begin{aligned}\hat{\chi}_S(\zeta)&=(1-i)^{n-1}\left(\frac{\pi}{2}\right)^{n-1}r^{(1-n)/2}\sum K(x^j)^{-1/2}\exp(-i\langle x^j,\zeta\rangle)\\&\quad+I_1+I_2+I_3;\\|I_1(\zeta)|&\leqq r^{-n/2}(1+|\eta|)^q\sum a_j\exp(\langle x^j,\eta\rangle),\\|I_2(\zeta)|&\leqq r^{-q}(1+|\eta|)^{2q}\sum b_j\exp[h_{S^j}(\eta)],\\|I_3(\zeta)|&\leqq c_\nu r^{-\nu}(1+|\eta|)\exp[h_{S^2}(\eta)],\end{aligned}\tag{4}$$

where $\sum=\sum_{j=0,1}$. Formula (4) combined with a result of Hörmander **[8]** yields Theorem 1 for $\chi_{\partial P}$. Asymptotic expansions similar to (4) hold for $\hat{\chi}_P$ as well as for the Fourier transforms of certain measures with non-constant density.

3. **The discrete case. Generalization of Ritt's theorem.** In this part we shall consider finite exponential sums, and more generally, exponential polynomials in several complex variables. If H is an exponential polynomial, i.e. a function of the form

$$H(\zeta)=\sum_{j=1}^{s}h_j(\zeta)\exp(\langle\theta_j,\zeta\rangle) \qquad (\zeta\in C^n)\tag{5}$$

with complex frequencies $\theta_j\in C^n$ and polynomial coefficients h_j, the greatest common divisor of the h_j's, $d_H=(h_1,\cdots,h_s)$, will be called the *content* of H. Moreover, we shall write $\mathfrak{C}_H(\zeta)=\max_j\operatorname{Re}\langle\theta_j,\zeta\rangle$. Henceforth an *exponential sum* will mean a function of the form (5) with all coefficients h_j constant. The following lower estimate of exponential polynomials was proved in **[3]**, **[5]**:

(R_0) Given an exponential polynomial H and an arbitrary $\varepsilon>0$, there exists $C=C(\varepsilon,H)>0$ such that for every $\zeta\in C^n$ and any f analytic in $\Delta(\zeta;\varepsilon)$,

$$|f(\zeta)|\exp(\mathfrak{C}_H(\zeta))\leqq C\,|f(\zeta)H(\zeta)|_\varepsilon.\text{[3]}\tag{6}$$

[3] Obviously, estimate (R_0) is much stronger than (R_ω).

In this section we shall discuss the following

Question. Let F and G be exponential polynomials in n variables such that the function $H=F/G$ is entire. What can be said about the structure of H? In particular, when is H an exponential polynomial?

Simple examples show that H need not be an exponential polynomial (e.g., $n=1$, $F=\sin\zeta$, $G=\zeta$). On the other hand, if F and G are exponential sums in one variable such that H is entire, then, according to a theorem of Ritt [12], H is also an exponential sum. Different proofs of Ritt's theorem were given by H. Selberg, P. D. Lax and A. Shields (cf. the references in [12], [13]). In particular, Shields [13] proves that H is an exponential polynomial as long as it is entire and G is an exponential sum. He also mentions that, according to an unpublished result of W. D. Bowsma, the last assumption may be replaced by $d_G=1$. Finally, Avanissian and Martineau [1] generalized the original Ritt's theorem to arbitrary $n>1$. The following theorem contains all these results as special cases. Moreover, it shows that the above counterexample is in a certain sense the best possible:

THEOREM 2. *Let F, G, H be as above ($n\geqq 1$ arbitrary). Then there exists an exponential polynomial E and a polynomial Q such that $H=E/Q$. Hence we may assume $(d_E, Q)=1$. Then E and Q are determined uniquely*[4] *and Q divides d_G.*

The starting point for the proof of Theorem 2 is the following assertion: Let f, g, h be the analytic functionals whose Fourier-Borel transforms are F, G, H respectively. Then h is carried by the polyhedron defined by $\mathfrak{C}_F-\mathfrak{C}_G$. This in turn follows from (R_0).

The proofs together with applications of the above theorems will appear elsewhere.

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