BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 3, May 1974

A CHARACTERIZATION OF THE FACTORS OF ORDINARY LINEAR DIFFERENTIAL OPERATORS

BY ANTON ZETTL

Communicated by Fred Brauer, October 8, 1973

Consider the ordinary differential operator L defined by

(1) $$Ly = y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_0 y \quad \text{for } y \in C^n(I)$$

where $p_i \in C^i(I)$ and I is any interval of the real line.

For $1 \leqq k < n$, let D_k denote the class of operators Q of type

$$Qy = y^{(k)} + q_{k-1}y^{(k-1)} + \cdots + q_0 y$$

with $q_i \in C^{n-k}(I)$ for $i=0, \cdots, k-1$.

By $W(y_1, \cdots, y_k)$ we mean the Wronskian of the class C^{k-1} functions $y_1, \cdots, y_k$, i.e. $W(y_1, \cdots, y_k) = \det[y_j^{(i-1)}]$.

In [4] it was shown that a necessary and sufficient condition for a factorization $L=RQ$ with $R \in D_{n-k}$, $Q \in D_k$ to hold is:

There exist solutions $y_1, \cdots, y_k$ of $Ly=0$ satisfying

(2) $$W(y_1, \cdots, y_k) \neq 0 \quad \text{on } I.$$

The factor Q has the form:

(3) $$Qy = W(y_1, \cdots, y_k, y)/W(y_1, \cdots, y_k) \quad \text{for all } y \in C^n.$$

Here we announce a characterization of R^*—the formal adjoint of the left factor R.

For a differential operator M denote by $N(M)$ the set of all solutions y of $My=0$.

Assume $y_1, \cdots, y_k$ are in $N(L)$ satisfying (2). Let $y_1, \cdots, y_k, \cdots, y_n$ be a basis of $N(L)$. Define

$$\bar{z}_i = W(y_1, \cdots, \hat{y}_i, \cdots, y_n)/W(y_1, \cdots, y_n) \quad \text{for } i = 1, \cdots, n$$

where the circumflex over y_i indicates that y_i is missing and $\bar{z}$ denotes the conjugate of the complex number z.

THEOREM. *Suppose a factorization $L=RQ$ with Q given by* (3) *holds. Then R is unique and*

(4) $$R^*z = W(z_{k+1}, \cdots, z_n, z)/W(z_{k+1}, \cdots, z_n) \quad \textit{for all } z \in C^n.$$

AMS (MOS) subject classifications (1970). Primary 34A30, 34A05; Secondary 34A01.

Key words and phrases. Ordinary linear differential equations, factoring linear differential operators, Wronskians, Lagrange bilinear form, conjugate solutions, formal adjoint operator.

Furthermore, given $z \in N(L^*)$, $[z, y_i]=0$ *for all* $i=1, \cdots, k$ *if and only if* $z \in N(R^*)$ *where* $[\ ,\ ]$ *is the Lagrange bilinear form associated with* L *i.e.*

$$[u, v] = \sum_{i=0}^{n} \sum_{j=0}^{i-1} (-1)^j (p_i \bar{v})^{(j)} u^{(i-1-j)}$$

for $u, v \in C^n(I)$.

COROLLARY. *Suppose* $n=2k$. *Then* $R^*=Q$ *if and only if* $z_j \in N(Q)$ *for all* $j=k+1, \cdots, n$.

The special case when L is formally selfadjoint and of order $2k$ reduces to the well-known result (see Heinz [**2**, Satz 3 and Zusatz p. 16], W. A. Coppell [**1**, Theorem 19, p. 80] and M. G. Krein [**3**]) that $L=Q^*Q$ with Q given by (3) if and only if there exist $y_1, \cdots, y_k$ in $N(L)$ which satisfy (2) and are pairwise conjugate, i.e. $[y_i, y_j]=0$ for all $i, j=1, \cdots, k$.

Since much more information is available about lower order operators than higher order ones—particularly for orders 2, 3 and 4—it is expected that the factorization $L=RQ$ will be useful by reducing the study of a problem to one of lower order. For example, we consider the study of disconjugacy.

It follows directly from the Pólya factorization of disconjugate operators that L is disconjugate if both R and Q are. It is also known [**1**] that R is disconjugate if R^* is. By applying Pólya's condition W to R^* and Q we obtain a disconjugacy criterion for L:

Functions $v_1, \cdots, v_p$ from C^{p-1} are said to have property W (or form a Markov system in the terminology of [**1**]) if the p Wronskians $W(v_1, \cdots, v_i)$ for $i=1, \cdots, p$ are positive.

The operator L is disconjugate if, for some k with $1 \leqq k < n$, there exist $y_1, \cdots, y_k \in N(L)$ such that $y_1, \cdots, y_k$ and some reordering of $z_{k+1}, \cdots, z_n$ have property W.

The proof is too long to be included here and will be published elsewhere together with some related results and applications and illustrations.

REFERENCES

1. W. A. Coppell, *Disconjugacy*, Lecture Notes in Math., vol. 220, Springer-Verlag, Berlin and New York, 1971.

2. E. Heinz, *Halbbeschränktheit gewöhnlicher Differentialoperatoren höherer Ordnung*, Math. Ann. **135** (1958), 1–49. MR **21** #743.

3. M. G. Krein, *Sur les operateurs differentiels auto-adjoints et leurs fonctions de Green symetriques*, Mat. Sb. (N.S.) **2** (44) (1937), 1023–1072.

4. A. Zettl, *Factorization of differential operators*, Proc. Amer. Math. Soc. **27** (1971), 425–426. MR **42** #7966.

DEPARTMENT OF MATHEMATICS, NORTHERN ILLINOIS UNIVERSITY, DEKALB, ILLINOIS 60115