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ON MEASURABILITY, POINTWISE CONVERGENCE AND COMPACTNESS[1]

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The starting point of this investigation is the beautiful generalization of Egorov's theorem given by P. A. Meyer in Séminaire de Probabilités. V, (Strasbourg). The material is divided as follows:

§1. Setting and terminology.
§2. The Generalized Egorov Theorem.
§3. An application to vector-valued mappings.
§4. The ⟨⟨separation property⟩⟩ and the notion of lifting.

Proofs of most of the results contained in this paper can be found in **[5]**, **[6]**, **[7]**, **[8]**.

1. **Setting and terminology.** Throughout this article $(E, \mathscr{E}, \mu)$ will be a fixed probability space. We denote by $\mathscr{L}=\mathscr{L}(E, \mathscr{E}, \mu)$ the algebra of all $f: E\to R$ which are $\mathscr{E}$-measurable.

For $f \in \mathscr{L}$, $g \in \mathscr{L}$ we write

$$f \equiv g \quad \text{if } f(t) = g(t) \quad \mu\text{-almost surely,}$$

and

$$f = g \quad \text{if } f(t) = g(t) \quad \text{for all } t \in E.$$

For $f\in \mathscr{L}$, we denote by $\tilde{f}$ the equivalence class of f with respect to the equivalence relation "$\equiv$" defined above.

We denote by $\mathscr{L}^\infty=\mathscr{L}^\infty(E, \mathscr{E}, \mu)$ the algebra of all *bounded* $f \in \mathscr{L}$.

For a set $B \in \mathscr{E}$ we denote by 1_B the indicator function of B (i.e. $1_B(t)=1$ for $t \in B$ and $1_B(t)=0$ for $t \in E-B$).

We say that a set $A \in \mathscr{E}$ *carries* μ if $\mu(E-A)=0$.

2. **The Generalized Egorov Theorem.** We may now state P. A. Meyer's generalization of Egorov's theorem (see **[8**, p. 199]) as follows:

THEOREM 1 (GENERALIZED EGOROV THEOREM). *Let $H\subset \mathscr{L}$ be compact and metrizable for the topology of pointwise convergence on E. There exists*

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then a sequence (A_n) *of disjoint subsets of* E *such that* $A_n \in \mathscr{E}$ *for each* n, *and* $\bigcup_n A_n$ *carries* μ, *with the following property*:

If (h_j) *is any sequence of elements of* H, *converging pointwise on* E, *say to* h, *then for each* n, $h_j|_{A_n}$ *converges uniformly to* $h|_{A_n}$.

REMARK 1. It should be stressed that the decomposition (A_n) is independent of the particular sequence (h_j) in H.

Let $H \subset \mathscr{L}$. We say that H satisfies the ⟨⟨*separation property*⟩⟩ if:

$$h_1 \in H,\ h_2 \in H,\ h_1 \neq h_2 \Rightarrow \tilde{h}_1 \neq \tilde{h}_2.$$

The relevant comment on the Generalized Egorov Theorem and the ⟨⟨*separation property*⟩⟩ is formulated in Theorem 2 below (see **[5]**; see also the notion of ⟨⟨partitionable function⟩⟩ introduced by M. Sion **[10**, p. 590]):

THEOREM 2. *Let* $H \subset \mathscr{L}$ *be compact metrizable for the topology of pointwise convergence on* E. *Then there is a set* $E_0 \in \mathscr{E}$ *carrying* μ, *with the following properties*:

(1) *For each* $\varepsilon > 0$ *there is a partition* (A_n^ε) *of* E_0 *with* $A_n^\varepsilon \in \mathscr{E}$ *and* $\mu(A_n^\varepsilon) > 0$ *for each* n, *such that*:

$$s \in A_n^\varepsilon,\ t \in A_n^\varepsilon,\ h \in H \Rightarrow |h(s) - h(t)| \leqq \varepsilon.$$

(2) $H|_{E_0}$ *satisfies the* ⟨⟨*separation property*⟩⟩.

We next make some remarks concerning the topology of pointwise convergence on a set of measurable functions:

REMARK 2. Let $H = \{h_1, h_2, \cdots, h_n, \cdots\}$ where each $h_n = 1_{B_n}$, with $B_n \in \mathscr{E}$. We may identify H with a subset of the compact space $\{0, 1\}^E$; the topology of pointwise convergence is then simply the product space topology. Various pathologies may occur:

(2.1) One may construct a sequence (h_n) such that *every* cluster value of this sequence is *non*-measurable (see for instance **[1**, Chapter IV (1952), p. 199, Exercise 4]).

(2.2) One may find $H = \{h_1, h_2, \cdots\}$ such that $\bar{H} = \{0, 1\}^E$.

Therefore the following questions are of interest:

Question 1. Let $H \subset \mathscr{L}$ be countable. Under what conditions is $\bar{H}$ compact metrizable (for the topology of pointwise convergence on E)?

Question 2. Let $H \subset \mathscr{L}$ be compact. Under what conditions is H metrizable (for the topology of pointwise convergence on E, of course)?

We shall begin with Question 2.

The following is a partial answer to Question 2, which however suffices for practical purposes (in Theorem 3 below we consider of course H endowed with the topology of pointwise convergence on E):

THEOREM 3 (METRIZATION CRITERION). *Let $H \subset \mathscr{L}$ be a set with the following properties*:

(i) *H is compact.*

(ii) *H is convex.*

(iii) *H satisfies the ⟨⟨separation property⟩⟩.*

Then H is metrizable.

To prove Theorem 3 one shows that, under our assumptions, the topology of pointwise convergence and the topology of convergence in probability coincide (see [6]). One may use in the proof the following remarkable theorem due to Komlós (see [7] or [2]):

THEOREM 4 (KOLMÓS). *Let (f_n) be a sequence of elements of $\mathscr{L}^1(E, \mathscr{E}, \mu)$ with $\sup_n \|f_n\|_1 < \infty$. Then one can find a subsequence (f_{n_k}) and an element $f \in \mathscr{L}^1(E, \mathscr{E}, \mu)$ such that (f_{n_k}), as well as any further subsequence extracted from (f_{n_k}), converges Cesàro to f, μ-almost surely.*

3. **An application to vector-valued mappings.** Let X be a *Banach space*, X' its *dual*. We denote the duality by $\langle x', x\rangle$, $x \in X$, $x' \in X'$. Let now $g: E \to X$. For $x' \in X'$ we denote by $\langle x', g\rangle$ the mapping $t \to \langle x', g(t)\rangle$ of E into R.

We recall that $g: E \to X$ is called *weakly measurable* if the real-valued mapping $\langle x', g\rangle$ is $\mathscr{E}$-measurable for each $x' \in X'$. We recall also that $g: E \to X$ is called *strongly* (Bochner) *measurable* if there is a sequence (s_n) of simple functions such that $\lim_n s_n(t) = g(t)$, μ-almost surely.

We may now state the following theorem (see [6]):

THEOREM 5 (WEAK VERSUS STRONG MEASURABILITY). *Let $g: E \to X$ be weakly measurable. We have*:

(1) *Suppose that the relations $x' \in X'$, $y' \in X'$ and $\langle x', g\rangle \neq \langle y', g\rangle$ imply $\langle x', g\rangle \not\equiv \langle y', g\rangle$. Then g is strongly measurable.*

(2) *Conversely, if $g: E \to X$ is strongly measurable, there is a set $E_0 \in \mathscr{E}$ carrying μ such that the relations $x' \in X'$, $y' \in X'$ and $\langle x', g\rangle|_{E_0} \neq \langle y', g\rangle|_{E_0}$ imply $\langle x', g\rangle \not\equiv \langle y', g\rangle$.*

It appears, therefore, that the ⟨⟨separation property⟩⟩ really makes the difference between weak measurability and strong measurability.

4. **The ⟨⟨separation property⟩⟩ and the notion of lifting.** The most *convenient* way to obtain the ⟨⟨separation property⟩⟩, at least for sets of bounded measurable functions, is by applying the *notion of lifting*:

We recall that a mapping $\rho: \mathscr{L}^\infty \to \mathscr{L}^\infty$ is called a *lifting of $\mathscr{L}^\infty$* if it satisfies the following conditions:

(I) $\rho(f)\equiv f$;

(II) $f\equiv g$ implies $\rho(f)=\rho(g)$;

(III) $\rho(1)=1$;

(IV) $\rho(af+bg)=a\rho(f)+b\rho(g)$;

(V) $\rho(fg)=\rho(f)\rho(g)$.

Without going into the history of the subject, it suffices to recall that if $(E, \mathscr{E}, \mu)$ is a *complete probability space*, then a lifting of $\mathscr{L}^\infty$ always exists (see for instance [**4**]).

Henceforth we assume that $(E, \mathscr{E}, \mu)$ is a complete probability space.

There is an equivalent way of defining the notion of lifting if one prefers to work with sets rather than functions. For each $A \in \mathscr{E}$, $\rho(1_A)$ is again an indicator function (by axiom (V)); we write

$$(*) \qquad \rho(1_A) = 1_{\rho(A)}.$$

The mapping $\rho:\mathscr{E}\to\mathscr{E}$ obtained in this manner satisfies the conditions:

(I′) $\rho(A)\equiv A$;

(II′) $A\equiv B$ implies $\rho(A)=\rho(B)$;

(III′) $\rho(E)=E$, $\rho(\varnothing)=\varnothing$;

(IV′) $\rho(A\cup B)=\rho(A)\cup\rho(B)$;

(V′) $\rho(A\cap B)=\rho(A)\cap\rho(B)$.

The mapping $\rho:\mathscr{E}\to\mathscr{E}$ satisfying axioms (I′)–(V′) is called a *lifting* of $\mathscr{E}$.

Since this cannot lead to confusion, we shall use the same notation for the lifting of $\mathscr{L}^\infty$ and the corresponding lifting of $\mathscr{E}$.

Lifting topology. Let now ρ be a fixed lifting of $\mathscr{L}^\infty=\mathscr{L}^\infty(E, \mathscr{E}, \mu)$. Corresponding to the lifting ρ we may introduce a topology $\mathscr{T}_\rho$ on the space E as follows:

$$\mathscr{T}_\rho = \{\rho(A) - N \mid A \in \mathscr{E}, N \in \mathscr{E}, \mu(N) = 0\}.$$

The topology $\mathscr{T}_\rho$ turns out to have the following properties (see [**4**, p. 59]):

(1) $\mathscr{T}_\rho$ is extremally disconnected.

(2) $C_R^b(E, \mathscr{T}_\rho)=\{\rho(g) | g \in \mathscr{L}^\infty\}$.

We may now give an answer to Question 1 raised in §2.

We shall only consider the case of a bounded set $H\subset\mathscr{L}^\infty$. We have the following analogue of Arzela-Ascoli's theorem (see [**5**], [**6**]):

THEOREM 6. *Let $H\subset\mathscr{L}^\infty$ be a bounded set. We have*:

(1) *Suppose that H is compact metrizable for the topology of pointwise convergence on E. There is then a set $E_0 \in \mathscr{E}$ carrying μ, such that $H|_{E_0}\subset C_R^b(E, \mathscr{T}_\rho)|_{E_0}$ and $H|_{E_0}$ is equicontinuous on (the $\mathscr{T}_\rho$-open set) E_0 with respect to $\mathscr{T}_\rho$.*

(2) *Conversely, suppose that $H\subset C_R^b(E, \mathscr{T}_\rho)$ and that H is equicontinuous*

with respect to $\mathscr{T}_\rho$. *Then* $\bar{H}$ *(closure of* H *for the topology of pointwise convergence* on E) *is compact metrizable.*

Another application. Let $(E, \mathscr{E}, \mu)$ be a complete probability space and Z a completely regular topological space.

We recall the definition of the *abstract space* $\mathscr{L}_Z^\infty = \mathscr{L}_Z^\infty(E, \mathscr{E}, \mu)$. A mapping $f: E \to Z$ belongs to $\mathscr{L}_Z^\infty$ if:

(i) $f(E) \subset Z$ is relatively compact;

(ii) $f: E \to Z$ is weakly measurable, that is, $h \circ f$ is $\mathscr{E}$-measurable, for each $h \in C_R(Z)$.

It is clear that if $Z = R$, then $\mathscr{L}_R^\infty = \mathscr{L}^\infty$.

Let now ρ be a *lifting of* $\mathscr{L}^\infty$. Starting with ρ one may define an ⟨⟨abstract lifting⟩⟩ of the abstract space $\mathscr{L}_Z^\infty$ as follows: For $f \in \mathscr{L}_Z^\infty$ we set

$$h \circ \rho_Z(f) = \rho(h \circ f), \quad \text{for all } h \in C_R(Z).$$

The above "weak invariance formula" uniquely determines the abstract lifting ρ_Z associated with ρ (see [**4**, pp. 52–53]). Since there can be no confusion, we shall denote this abstract lifting by ρ again.

This notion of abstract lifting has many advantages: let us mention in passing that it permits to give a very simple and rapid proof of Doob's classical theorem concerning the "existence of a separable modification" of a stochastic process (see [**3**] or [**4**]).

Let us now consider again a Banach space X and let us return to weakly measurable versus strongly measurable mappings.[2]

Consider $(X, \sigma(X, X'))$ and correspondingly the abstract space $\mathscr{L}^\infty_{(X,\sigma(X,X'))}$.

We note that $f \in \mathscr{L}^\infty_{(X,\sigma(X,X'))}$ if

(i) $f(E) \subset X$ is $\sigma(X, X')$-relatively compact.

(ii) $\langle x', f\rangle$ is $\mathscr{E}$-measurable, for each $x' \in X'$.

We then have (see [**5**]):

THEOREM 7. *Let* $f \in \mathscr{L}^\infty_{(X,\sigma(X,X'))}$ *and let* $g = \rho(f)$. *Then*

$$\rho(\langle x', g\rangle) = \langle x', g\rangle, \quad \textit{for each } x' \in X'$$

and hence g *is strongly measurable.*

It is enough to note that since g satisfies the above invariance property, g has the separation property required in Theorem 5, and therefore g is strongly measurable.

[2] Below we denote by $\sigma(X, X')$ the X'-topology on X, that is the weakest topology on X making every linear functional $x' \in X'$ continuous.

REMARK 3. The fact that if $f: E \to X$ is a weakly measurable mapping taking values in a $\sigma(X, X')$-compact set, then its $\langle\langle$weak equivalence class$\rangle\rangle$ contains a strongly measurable mapping is the object of a classical theorem due to R. S. Phillips [**9**]. The known proofs introduce the associated weakly compact operator of L^1_R into X (see [**1**, Chapter 6 (1959), p. 95, Exercise 25], or [**4**, pp. 91–92]), or the associated vector-valued measure (see [**11**, pp. 115–118]) and are quite laborious. The separation property, as exhibited in Theorem 5, makes the notion of lifting appear as a natural tool in this type of problem: We obtain a one-line proof of Phillips' theorem. Furthermore the notion of lifting yields a canonical way of constructing a strongly measurable function g in the weak equivalence class of f.

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