BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 6, November 1973

ASYMPTOTIC THEOREMS FOR SUMS OF INDEPENDENT RANDOM VARIABLES DEFINED ON A TREE[1]

BY A. JOFFE AND A. R. MONCAYO

Communicated by Gian-Carlo Rota, December 18, 1972

The study of sums of independent random variables defined on a tree has not been treated systematically in the literature, except for the random tree generated by a Galton-Watson process (cf. [1], [4], [5]) and for the binary tree (cf. [3]). The purpose of this short note is to announce a generalization of the results of the above papers.

1. A tree $\mathscr{T}$ will be here a collection of sequences $\tau = (i_1 \cdots i_k \cdots)$ where the i_j are nonnegative integers such that

(a) if $i_l = 0$, then $i_k = 0$ for all $k > l$.

(b) $i_1 = 1 \cdots Z_1$.

(c) for $k > 1$, $i_k = 1 \cdots Z_{i_1 \cdots i_{k-1}}$ or 0, $\sum_{i_1 \cdots i_{k-1}} Z_{i_1 \cdots i_{k-1}} = Z_k$. We require $Z_k \geqq 1$.

Given a tree $\mathscr{T}$, we define $\mathscr{T}_k$, the family of size k of $\mathscr{T}$, to be the set of finite sequences $\tau_k = (i_1 \cdots i_k)$ of length k which are the beginning of a sequence of the tree such that $i_k \neq 0$. The cardinality of $\mathscr{T}_k$ is Z_k. We denote by $\alpha(n, k)$ the number of ordered pairs of the path of $\mathscr{T}_n$ which have exactly in common an initial path of length k. Let $p_{n,k} = \alpha(n, k)/Z_n^2$; we say that the tree is regular if $\lim_{n\to\infty} p_{n,k} = p_k$ exists with $\sum_k p_k = 1$. Let g be a nonnegative nondecreasing function defined on the integers.

(a) We say that the tree $\mathscr{T}$ is g-regular if it is regular and if

$$\lim_{n\to\infty} \sum_{k=0}^{n} g(k)p(n, k) = \sum_{k=0}^{\infty} g(k)p_k < \infty.$$

(b) We say that the tree $\mathscr{T}$ is weakly g-regular if

$$\sup_n \sum_{k=0}^{n} g(k)p(n, k) < \infty.$$

We consider a family of independent identically distributed random variables X_{τ_k} indexed by $\bigcup_{k=1}^{\infty} \mathscr{T}_k$. To simplify notations and the statement of the theorems we assume the X's to have mean 0 and variance 1. At each path $\tau_n = (i_1 \cdots i_n)$ we associate the random variables

---

AMS (MOS) subject classifications (1970). Primary 60B10, 60J80.

Key words and phrases. Sums of independent random variables, trees, Galton-Watson.

[1] This research was supported by CIMASS (UNAM) and the Canadian National Research Council.

$$S_{\tau_n} = X_{i_1} + X_{i_1 i_2} + \cdots + X_{i_1 \cdots i_n} \quad \text{and} \quad S^*_{\tau_n} = S_{\tau_n/n^{1/2}}.$$

The finite set $\{S^*_{\tau_n}\}_{\tau_n \in \mathscr{T}_n}$ defines a random point distribution on the real line. By assigning to each point the weight $1/Z_n$ we define a random probability measure

$$\mu_n(B) = \frac{1}{Z_n} \sum_{\tau_n} V_B(S^*_{\tau_n})$$

where V_B denotes the characteristic function of the Borel set B. We have studied the asymptotic properties of $\mu_n(B)$ and $\Psi_n(t)$ its random Fourier transform.

2. **Statement of results.** Let Φ be the measure whose density with respect to Lebesgue measure is the gaussian function $(2\pi)^{-1/2}e^{-x^2/2}$.

Let $c_k(n) = \sum_{j=k}^n jp(n, k)/n$ and for any $\varepsilon > 0$ let

$$d_\varepsilon(n, k) = P(|X_1 + \cdots + X_k| > n^{1/2}\varepsilon).$$

We have obtained the following theorems.

THEOREM 1. *A necessary and sufficient condition for the convergence in mean square of* Ψ_n *to* $e^{-t^2/2}$ *is that* $\sum_{k=1}^n kp(n, k) = o(n)$ *as n goes to infinity.*

THEOREM 2. *Let* $\mathscr{T}$ *be a weakly g-regular tree with* $g(k) = k$. *If there is an increasing sequence* k_n *such that*

(a) $\sum_n c_{k_n}(n) < \infty$,

(b) $\lim_{n\to\infty} k_n/n = 0$,

(c) $\lim_{n\to\infty} (Z_1 + \cdots + Z_{k_n})/n = 0$,

then μ_n *converges weakly to* Φ *with probability one. In particular the above conditions will be satisfied if* $\mathscr{T}$ *is weakly g-regular with* $g(k) = k^{1+\alpha}$, *with* $\alpha > 0$ *and if* (c) *is satisfied with* $k_n \geqq \mathrm{c} \log^\beta n$ *for some* β *larger than* $1/\alpha$.

THEOREM 3. *If the tree* $\mathscr{T}$ *and the random variables* X_{τ_k} *are such that one can find a sequence* $k_n \uparrow \infty$ *satisfying*

(a) $\sum_n c_{k_n}(n) < \infty$,

(b) $\sum d_\varepsilon(n, k_n) < \infty$, *for all* $\varepsilon > 0$,

(c) $\lim_{n\to\infty} k_n/n = 0$,

then μ_n *converges weakly to* Φ *with probability one. In particular the above conditions are satisfied if* $EX^2 \log^\gamma |X| < \infty$ *with* $\gamma > 2$ *and if* $\mathscr{T}$ *is weakly regular with* $g(k) = k^{1+\alpha}$ *where* $\alpha > 6/(\gamma - 2)$.

The proofs will appear elsewhere; the main tools are the use of the Fourier transform $\Psi_n(t)$ and the exploitation of the decomposition

$$\Psi_n - e^{-t^2/2} = (\Psi_n - E(\Psi_n \mid \mathscr{F}_k)) + (E(\Psi_n \mid \mathscr{F}_k)) - e^{t^2/2},$$

where $\mathscr{F}_k$ denotes the σ-field generated by X_{τ_l}, $\tau_l \in \bigcup_{j=1}^k \mathscr{T}_j$.

3. **Application to Galton-Watson process.** We consider random trees generated by a supercritical Galton-Watson process with finite variance. To avoid having to condition on nonextinction we will assume that extinction occurs with zero probability. It is easily seen that the random tree of the above process satisfies the assumptions of Theorem 2 with probability one. This strengthens the results of [4] and [5] and proves the conjecture stated in [2]. Moreover one can easily verify that the $p(n, k)$ form a martingale for k fixed and $n > k + 1$.

References

1. T. E. Harris, *The theory of branching processes*, Die Grundlehren der math. Wissenschaften, Band, 119, Springer-Verlag, Berlin; Prentice-Hall, Englewood Cliffs, N.J., 1963. MR **29** #664.

2. A. Joffe, *Branching processes which perform a random walk*, Symposium on Applied Stochastic Processes, Rochester, August 1971.

3. A. Joffe and A. R. Moncayo, *On sums of independent random variables defined on a binary tree*, Bol. Soc. Mat. Mexicana (to appear).

4. P. E. Ney, *The limit distribution of a binary cascade process*, J. Math. Anal. Appl. **10** (1965), 30–36. MR **30** #1551.

5. ———, *The convergence of a random distribution function associated with branching processes*, J. Math. Anal. Appl. **12** (1965), 316–327. MR **32** #1760.

Département de Mathématiques, Université de Montréal, Case Postale 6128, Montréal, Québec, Canada

CIMASS, Universidad Nacional Autonoma de Mexico, Apartado Postal 20-726, Mexico 20, D.F.