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THE GREEN FUNCTION OF A LINEAR DIFFERENTIAL EQUATION WITH A LATERAL CONDITION

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Let E be a Banach space. We consider systems of the form

$$\text{(L)} \quad L[y] \equiv y' + Ay = f, \qquad \text{(F)} \quad F[y] = c$$

where $y \in \mathscr{C}^{(1)}([a, b], E)$, $f \in \mathscr{C}([a, b], E)$, $A \in \mathscr{C}([a, b], L(E))$, $F \in L[\mathscr{C}([a, b], E), E]$ and $c \in E$. When the system has one and only one solution, for any $f \in \mathscr{C}([a, b], E)$ and $c \in E$, we show that it has a Green function, that is, a function $G:[a, b] \times [a, b] \to L(E, E'')$ such that $y \in \mathscr{C}^{(1)}([a, b], E)$ is the solution of $L[y] = f$ and $F[y] = 0$ if and only if $y(t) = \int_a^b G(t, s)f(s)\, ds$. We exhibit the relations between G, A and F. (F) is called a *lateral condition*; initial conditions and boundary conditions are particular instances of lateral conditions. The construction of G uses a Riemann-Stieltjes integral representation for F, given in §1.

1. **Analytic preliminaries.** We consider always vector spaces over the complex field $\boldsymbol{C}$, but all results are valid for real vector spaces.

1. Given an interval $[a, b]$ of the real line, a *division* of $[a, b]$ is a finite sequence $d: t_0 = a < t_1 < \cdots < t_n = b$. We write $|d| = n$ and $\Delta d = \sup\{|t_i - t_{i-1}| \mid i = 1, 2, \ldots, |d|\}$; D denotes the set of all divisions of $[a, b]$.

2. Let X, Y be Banach spaces; given $\alpha: [a, b] \to L(X, Y)$ and $d \in D$ we define

$$SV_d[\alpha] = \sup\left\{ \left\| \sum_{i=1}^{|d|} [\alpha(t_i) - \alpha(t_{i-1})]x_i \right\| \,\middle|\, x_i \in X, \|x_i\| \leqq 1 \right\}$$

and $SV[\alpha] = \sup\{SV_d[\alpha] | d \in D\}$.

We say that α is of *bounded semivariation*, and we write

$$\alpha \in SV([a, b], L(X, Y)),$$

if $SV[\alpha] < \infty$ (see for instance [**D**] and [**B-K**]).

PROPOSITION 1. *Given $\alpha \in SV([a, b], L(X, Y))$ and $f \in \mathscr{C}([a, b], X)$, there exists $F_\alpha[f] = \int_a^b d\alpha(t) \cdot f(t) = \lim_{\Delta d \to 0} \sum_{i=1}^{|d|} [\alpha(t_i) - \alpha(t_{i-1})] \cdot f(\xi_i) \in Y$, where $\xi_i \in [t_{i-1}, t_i]$, $i = 1, 2, \ldots, |d|$. We have $\|F_\alpha[f]\| \leqq SV[\alpha]\|f\|$ and hence*

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$F_\alpha \in L[\mathscr{C}([a,b],X),Y]$ *with* $\|F_\alpha\| \leqq SV[\alpha]$.

EXAMPLE 1. If $Y = \boldsymbol{C}$ then $SV([a,b],L(X,\boldsymbol{C})) = BV([a,b],X')$ where $BV([a,b],X')$ is the space of functions $\alpha:[a,b] \to X'$ that are of *bounded variation*, that is, such that $V[\alpha] = \sup\{V_d[\alpha] | d \in D\} < \infty$ where

$$V_d[\alpha] = \sum_{i=1}^{|d|} \|\alpha(t_i) - \alpha(t_{i-1})\|$$
$$= \sup\left\{\left|\sum_{i=1}^{|d|} \langle x_i, \alpha(t_i) - \alpha(t_{i-1})\rangle\right| \,\middle|\, x_i \in X, \|x_i\| \leqq 1\right\}.$$

By $B\widetilde{V}_0([a,b],X')$ we denote the space of all functions $\alpha \in BV([a,b],X')$ such that $\alpha(a) = 0$ and $\alpha(t+) = \alpha(t)$ for $t \in \,]a,b[$. Endowed with the norm $V[\alpha]$, $B\widetilde{V}_0([a,b],X')$ is a Banach space. We write

$$B\widetilde{V}_0([a,b]) = B\widetilde{V}_0([a,b],\boldsymbol{C}).$$

In the usual way one proves the following

THEOREM 2 (RIESZ). $\mathscr{C}([a,b],X)' \cong B\widetilde{V}_0([a,b],X')$; *i.e., the mapping* $\alpha \in B\widetilde{V}_0([a,b],X') \mapsto F_\alpha \in \mathscr{C}([a,b],X)'$ *is a linear isometry* (*i.e.,* $\|F_\alpha\| = V[\alpha]$) *of the first Banach space onto the second.*

EXAMPLE 2. If $X = \boldsymbol{C}$ we have $SV([a,b],L(\boldsymbol{C},Y)) = BW([a,b],Y)$, where $BW([a,b],Y)$ is the space of functions $\alpha:[a,b] \to Y$ that are of *weak bounded variation*, that is, $W[\alpha] = \sup\{W_d[\alpha] | d \in D\} < \infty$ where

$$W_d[\alpha] = \sup\left\{\left\|\sum_{i=1}^{|d|} \lambda_i[\alpha(t_i) - \alpha(t_{i-1})]\right\| \,\middle|\, \lambda_i \in \boldsymbol{C}, |\lambda_i| \leqq 1\right\}.$$

DEFINITION. Let Z be a Banach space;

$B\widetilde{W}_0([a,b],Z') = \{\alpha \in BW([a,b],Z') | z \circ \alpha \in B\widetilde{V}_0([a,b])$ for every $z \in Z\}$.

Endowed with the norm $W([\alpha]$, $B\widetilde{W}_0([a,b],Z')$ is a Banach space.

THEOREM 3. $L[\mathscr{C}([a,b]),Z'] \cong B\widetilde{W}_0$

$$\alpha \in B\widetilde{W}_0([a,b],Z') \mapsto F_\alpha \in L[\mathscr{C}([a,b]),Z']$$

is a linear isometry (*that is* $\|F_\alpha\| = W[\alpha]$) *of the first Banach space onto the second* (*for* $\phi \in \mathscr{C}([a,b])$ *we define* $F_\alpha[\phi] = \int_a^b \phi(t)\,d\alpha(t)$).

3. Let X and Z be Banach spaces and $\alpha:[a,b] \to L(X,Z')$. Given $x \in X$ and $z \in Z$ we define $\alpha(x):[a,b] \to Z'$ and $(z \circ \alpha)(x):[a,b] \to \boldsymbol{C}$ by $\langle z, \alpha(x)(t)\rangle = \langle z, \alpha(t)x\rangle$ and $(z \circ \alpha)(x)(t) = \langle z, \alpha(t)x\rangle$.

We have

$$BV([a,b],L(X,Z')) \subset SV([a,b],L(X,Z')) \subset BW([a,b],L(X,Z'))$$

and we define

$\widetilde{SV}_0([a,b], L(X,Z'))$

$= \{\alpha \in SV([a,b], L(X,Z')) | (z \circ \alpha)(x) \in \widetilde{BV}_0([a,b])$ for all $x \in X$ and $z \in Z\}$.

Endowed with the norm $SV[\alpha]$, $\widetilde{SV}_0([a,b], L(X,Z'))$ is a Banach space.

The following theorem is an extension of Riesz' representation theorems (Theorems 2 and 3):

THEOREM 4. *Let X and Z be Banach spaces. The mapping*

$$\alpha \in \widetilde{SV}_0([a,b], L(X,Z')) \mapsto F_\alpha \in L[\mathscr{C}([a,b],X), Z']$$

is a linear isometry (i.e. $\|F_\alpha\| = SV[\alpha]$) from the first Banach space onto the second (see for instance [**B-K**, Satz 11]).

COROLLARY. *Let X and Y be Banach spaces. For every*

$$F \in L[\mathscr{C}([a,b],X), Y]$$

there is one and only one $\alpha \in \widetilde{SV}_0([a,b], L(X, Y''))$ such that $F = F_\alpha$; we write $\alpha_F = \alpha$.

4. In what follows we extend the preceding results to locally convex topological vector spaces (LCTVS). We do not use these extended results in this paper.

Let X and Y be LCTVS; we denote by P and Q the set of all continuous seminorms defined on X and Y, respectively. Given $\alpha : [a,b] \to L(X,Y)$, $p \in P$, $q \in Q$ and $d \in D$, we define

$$SV_{q,p;d}[\alpha] = \sup\left\{ q\left[\sum_{i=1}^{|d|} [\alpha(t_i) - \alpha(t_{i-1})] \cdot x_i\right] \middle| x_i \in X, p(x_i) \leqq 1\right\}$$

and $SV_{q,p}[\alpha] = \sup\{SV_{q,p;d}[\alpha] | d \in D\}$. We write $\alpha \in SV_{q,p}([a,b], L(X,Y))$ if $SV_{q,p}[\alpha] < \infty$. We say that α is of *bounded semivariation*, and we write $\alpha \in SV([a,b], L(X,Y))$, if for every $q \in Q$ there is a $p \in P$ such that $SV_{q,p}[\alpha] < \infty$; that is $SV([a,b], L(X,Y)) = \bigcap_{q \in Q}[\bigcup_{p \in P} SV_{q,p}([a,b], L(X,Y))]$.

PROPOSITION 1′. *Let X and Y be LCTVS, Y sequentially complete. Given $\alpha \in SV([a,b], L(X,Y))$ and $f \in \mathscr{C}([a,b], Y)$ there exists $F_\alpha[f] = \int_a^b d\alpha(t) \cdot f(t) = \lim_{\Delta d \to 0} \sum_{i=1}^{|d|} [\alpha(t_i) - \alpha(t_{i-1})] \cdot f(\xi_i) \in Y$, where $\xi_i \in [t_{i-1}, t_i]$. We have $F_\alpha \in L[\mathscr{C}([a,b],X), Y]$.*

If $Y = \boldsymbol{C}$ we get $SV([a,b], L(X,\boldsymbol{C})) = BV([a,b], X')$ where $BV([a,b], X')$ is the space of functions $\alpha : [a,b] \to X'$ that are of *bounded variation*, i.e., $BV([a,b], X') = \bigcup_{p \in P} BV_p([a,b], X')$; $BV_p([a,b], X')$ denotes the space of functions $\alpha : [a,b] \to X'$ such that $V_p[\alpha] < \infty$, where

$$V_p[\alpha] = \sup\{V_{p;d}[\alpha] | d \in D\}$$

and

$$V_{p,d}[\alpha] = \sup\left\{ \left| \sum_{i=1}^{|d|} \langle x_i, \alpha(t_i) - \alpha(t_{i-1})\rangle \right| \, \middle| \, x_i \in X, p(x_i) \leqq 1 \right\}.$$

By $\widetilde{BV}_0([a,b], X')$ we denote the space of all functions $\alpha \in BV([a,b], X')$ such that $x \circ \alpha \in \widetilde{BV}_0([a,b])$ for all $x \in X$.

Using the classical Riesz theorem (Theorem 2) we prove

THEOREM 2'. *$\mathscr{C}([a,b], X)' \cong \widetilde{BV}_0([a,b], X')$; i.e., the mapping*

$$\alpha \in \widetilde{BV}_0([a,b], X') \mapsto F_\alpha \in \mathscr{C}([a,b], X)'$$

is an isomorphism of the first vector space onto the second.

Let X and Z be LCTVS, Z bornological; by Z'_b we denote its topological dual endowed with the strong topology. We define $\widetilde{SV}_0([a,b], L(X, Z'_b)) = \{\alpha \in SV([a,b], L(X, Z'_b)) | z \circ \alpha \in \widetilde{BV}_0([a,b], X')$ for all $z \in Z\}$.

Using Theorem 2' we prove

THEOREM 4'. *Let X and Z be LCTVS, Z bornological. The mapping*

$$\alpha \in \widetilde{SV}_0([a,b], L(X, Z'_b)) \mapsto F_\alpha \in L[\mathscr{C}([a,b], X), Z'_b]$$

is an isomorphism of the first vector space onto the second.

Endowing the spaces above with their natural structure of LCTVS the algebraic isomorphisms in Theorems 2' and 4' become homeomorphisms.

2. **The Green function.** 1. Given the differential operator L defined in the introduction we denote by R_s its resolvent, i.e., for every $s \in [a,b]$, R_s is the solution $R \in \mathscr{C}^{(1)}([a,b], L(E))$ of $dR/dt + A \circ R = 0$ such that $R(s) = I_E$ (identical automorphism of E). We write $R(t,s) = R_s(t)$, where $t \in [a,b]$.

THEOREM 5. *The solution of $L[y] = f$, $y(s) = c$ is given by $y(t) = R(t,s)c + \int_s^t R(t,\sigma) f(\sigma)\, d\sigma$. (See, for instance,* [**B**] *or* [**C**].)

2. Given $F \in L[\mathscr{C}([a,b], E), E]$ and $s \in [a,b]$, for every $x \in E$ we define $F[R_s]x = F[R_s x]$, hence $F[R_s] \in L(E)$. It is easy to show that $F[R_s] = \int_a^b d\alpha(t) \circ R(t,s)$, where $\alpha = \alpha_F$. We write $J_s = J(s) = F[R_s] = F_t[R(t,s)]$.

The following theorem is easy to prove:

THEOREM 6. *The following properties are equivalent:*

(1) *For every $f \in \mathscr{C}([a,b], E)$ and $c \in E$, the system $L[y] = f$, $F[y] = c$ has one and only one solution $y \in \mathscr{C}^{(1)}([a,b], E)$.*

(2) *For every $c \in E$ the system $L[y] = 0$, $F[y] = c$ has one and only one solution $y \in \mathscr{C}^{(1)}([a,b], E)$.*

(3) *The mapping* $y \in \{u \in \mathscr{C}^{(1)}([a, b], E) | L[u] = 0\} \mapsto F[y] \in E$ *is an isomorphism of the first space onto the second.*

(4) *For every* $s \in [a, b]$ *we have* $J_s = F[R_s] \in \mathrm{Aut}(E)$.

(5) *There is an* $s \in [a, b]$ *such that* $J_s \in \mathrm{Aut}(E)$.

From now on we suppose that the equivalent properties of Theorem 6 *are verified.*

It is immediate that

(1) For every $t, s \in [a, b]$ we have $R(t, s) = J(t)^{-1} \circ J(s)$.

(2) $F_t[J(t)^{-1}] = I_E$.

(3) $dJ(t)^{-1}/dt = A(t) \circ J(t)^{-1} = 0$.

Using (3) one can prove that

(4) For every $f \in \mathscr{C}([a, b], E)$ there exist the following integrals and we have

$$\int_a^b d\alpha(\tau) \circ J(\tau)^{-1} \left[\int_a^\tau J(s) f(s)\, ds \right] = \int_a^b \left[\int_s^b d\alpha(\tau) \circ J(\tau)^{-1} \right] J(s) f(s)\, ds.$$

THEOREM 7. *If the properties of Theorem* 6 *are verified then*

$$y \in \mathscr{C}^{(1)}([a, b], E)$$

is the solution of the system $L[y] = f$, $F[y] = c$ *if and only if*

$$\text{(G)} \qquad y(t) = J(t)^{-1} c + \int_a^b G(t, s) f(s)\, ds$$

where

$$G(t, s) = \hat{J}(t)^{-1} \circ \left[\int_a^s d\alpha_F(\tau) \circ J(\tau)^{-1} - Y(s - t) I_E \right] \circ J(s) \cdot$$

$\hat{J}(t)^{-1} \in L(E'')$ *being the bitranspose of* $J(t)^{-1} \in L(E)$ *and* Y *the Heaveside function. We have*

(i) $G(t, s) \in L(E, E'')$;

(ii) $G(s+, s) - G(s-, s) = I_E$ *for every* $s \in]a, b[$;

(iii) $G(t, b) = 0$; $G(a, a) = -I_E$ *and* $G(t, a) = 0$ *for* $a < t \leqq b$;

(iv) *for every fixed* $s \in [a, b]$, G *is a continuous function of* t, *for* $t \neq s$;

(v) *for every fixed* $t \in [a, b]$ *and every* $x \in E$, *the function* $s \in]a, b[\mapsto G(t, s) \cdot x \in E''_{\sigma(E'', E')}$ *is continuous on the right*;

(vi) *the function* G *with these properties is unique.*

SKETCH OF THE PROOF. If $y \in \mathscr{C}^{(1)}([a, b], E)$ is the solution of the system $L[y] = f$, $F[y] = c$ by Theorem 5 and (1) we have

$$y(\tau) = R(\tau, t)y(t) + \int_t^\tau R(\tau, s)f(s)\,ds$$
$$= J(\tau)^{-1}J(t)y(t) + J(\tau)^{-1}\left[\int_a^\tau J(s)f(s)\,ds - \int_a^t J(s)f(s)\,ds\right].$$

Applying F and using (2), the corollary of Theorem 4 and (4), one proves that

$$c = J(t)y(t) - \int_a^b \left[\int_a^s d\alpha(\tau)\circ J(\tau)^{-1}\right] J(s)f(s)\,ds + \int_t^b J(s)f(s)\,ds,$$

from which (G) follows easily. Properties (i) to (v) follow from the expression for G and the proof of (vi) uses Theorem 8 below.

3. **Extensions of Theorem 7.** Theorem 7 may be adapted to the case in which the system $L[y] = f, F[y] = 0$ has one and only one solution for every $f \in \mathscr{C}([a, b], E)$. In this case $J_t^{-1}: E_0 = F[\mathscr{C}^{(1)}([a, b], E)] \to E$ is continuous if and only if E_0 is a closed subspace of E.

THEOREM 8. *The system $L[y] = f$, $F[y] = c$ where $f \in L_1([a, b], E)$ has one and only one solution $y \in L_1^{(1)}([a, b], E)$, given by* (G) *(but now the integral is defined by continuous extension of* (G) *from $\mathscr{C}([a, b], E)$ to $L_1([a, b], E)$).*

THEOREM 9. *If the system $L[y] = f$, $\hat{F}[y] = \hat{c}$, has one and only one solution $y \in \mathscr{C}^{(1)}([a, b], E)$ for every $f \in \mathscr{C}([a, b], E)$ and $\hat{c} \in E$, where $\hat{F} \in L[\mathscr{C}^{(1)}([a, b], E), E]$, then we can reduce it to a system $L[y] = f$, $F[y] = c$ that has also one and only one solution, where $F \in L[\mathscr{C}([a, b], E), E]$. F and c are given by*

$$F[y] = \hat{F}_t\left[y(a) - \int_a^t A(s)y(s)\,ds\right] \quad \textit{and} \quad c = \hat{c} - \hat{F}_t\left[\int_a^t f(s)\,ds\right].$$

The preceding results may be extended to systems of the form

$$L[y] \equiv A_0(A_1 y)' + By = f, \qquad F[y] = c$$

where $A_0, A_1, B \in \mathscr{C}([a, b], L(E))$ and A_0, A_1 are invertible at every point $t \in [a, b]$. In this case $y \in D_L = \{u \in \mathscr{C}([a, b], E) | A_1 u \in \mathscr{C}^{(1)}([a, b], E)\}$; D_L is endowed with the norm $\|u\|^{(L)} = \sup[\|u\|, \|(A_1 u)'\|]$ and $F \in L[D_L, E]$.

ADDED IN PROOF. The results of this paper may also be extended to half-open and open intervals, to the case where $A \in L_1^{\mathrm{loc}}(]a, b[, L(E))$ and F takes values in a Banach space different from E. The proofs will appear in [**H**].

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