

TOPOLOGIES ON SPACES OF BAIRE MEASURES

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Communicated by R. Creighton Buck, November 6, 1972

Introduction. Let X be a completely regular, Hausdorff space, let C be the space of real-valued continuous functions on X and let C^b be the subspace of C consisting of the uniformly bounded continuous functions on X . The Banach dual of C^b (for the uniform norm) will be denoted by M , and the subspace of M consisting of all totally-finite, signed Baire measures on X will be denoted by M_σ . (Recall that the algebra of Baire sets is the smallest σ -algebra of subsets of X for which each of the functions in C is measurable.) Finally, the space of signed measures in M_σ which have finite support will be denoted by L . By identifying each point of X with the Dirac measure at that point we may assume that X is a subset of L (and hence of M_σ). The purpose of the present note is to describe some results recently obtained by the author concerning completions of L relative to certain natural locally convex topologies on L , and some applications of these results. (For the proofs and for further details, the reader is referred to [5] and [6].) The principal results are essentially generalizations to arbitrary spaces of the following theorem due to M. Katětov and V. Pták. (See [3], [4] and [8].)

THEOREM. *Let X be pseudocompact. Then the completion of L for the topology of uniform convergence on the pointwise bounded, equicontinuous subsets of C is the space $M_\sigma (= M)$.*

In order to avoid certain technical difficulties in the discussion, we will assume throughout the paper that X has a nonmeasurable cardinal. (As is well known, it is consistent with the axioms of set theory to assume that there are no measurable cardinals.) For a discussion of the results in the presence of measurable cardinals, the reader is referred to [5] and [6].

1. **The topology e^b .** A set $B \subset C$ is *equicontinuous* if for all $x \in X$ and for every positive number ε , there is a neighborhood U of x such that $|f(x) - f(y)| \leq \varepsilon$ for all $y \in U$ and all $f \in B$. The set B is *uniformly bounded* if there is a number K such that $|f(x)| \leq K$ for all $f \in B$ and all $x \in X$. Let \mathcal{E}^b denote the family of all uniformly bounded, equicontinuous subsets of C^b ; and let e^b denote the topology on M_σ of uniform convergence on the sets in \mathcal{E}^b . It is easily verified that e^b is a locally convex topology on M_σ . We then have the following result:

AMS (MOS) subject classifications (1970). Primary 46A30, 60B10; Secondary 60B05.
Key words and phrases. Baire measures, weak convergence of Baire measures, complete spaces of Baire measures.

THEOREM 1.1. *Let X be a completely regular Hausdorff space, and let M_σ be endowed with the topology e^b . Then the following hold:*

1. M_σ is complete.
2. L is dense in M_σ .
3. The dual space of M_σ is C^b .
4. X is a bounded subset of M_σ .
5. The restriction of the topology e^b to X is the original topology on X .

The proof of the theorem can be outlined as follows. One first considers the space L with the topology e^b . It is not difficult to show that the dual of (L, e^b) is C^b , and that, in fact, e^b is the finest locally convex topology on L whose restriction to X is the original topology and for which X is a bounded subset. One then shows that the completion of (L, e^b) is (M_σ, e^b) from which the theorem follows easily. This is done by combining the well-known theorem of A. Grothendieck characterizing complete locally convex topologies [7, p. 270] with certain results from the theory of measures in topological spaces.

As an application of Theorem 1.1, the following result of E. Granirer [2] may be obtained. (Of course, M_σ^+ denotes the space of all finite Baire measures on X .)

THEOREM 1.2. *Let X be a completely regular Hausdorff space. Then the following hold:*

1. On M_σ^+ the topologies e^b and $\sigma(M_\sigma, C^b)$ are identical.
2. If B is a uniformly bounded, equicontinuous subset of C^b , then B is relatively $\sigma(C^b, M_\sigma)$ -compact.

It is clear that statement 2 in Theorem 1.2 is an immediate consequence of statement 3 in Theorem 1.1 and the Mackey-Arens theorem. The proof of statement 1 is somewhat more complicated. However, by using Theorem 1.1, the analysis can be done essentially on the measures which have finite support. In this way a more elementary proof than that given by Granirer is obtained.

2. The topology e . A set $B \subset C$ is *pointwise bounded* if $\sup\{|f(x)|: f \in B\}$ is finite for all $x \in X$. Let \mathcal{E} denote the family of all pointwise bounded, equicontinuous subsets of C . Let M_c denote the subspace of M_σ consisting of all those signed measures whose total variations have compact support in the realcompactification of X . (A Baire measure m is said to have compact support in the realcompactification νX of X if there is a compact set G in νX such that $m(X - Z) = 0$ for every zero set Z in νX with $G \subset Z$.) Then M_c and C may be paired by the bilinear form $\langle m, f \rangle = \int_X f dm = \int_X f dm^+ - \int_X f dm^-$. It can be verified that the topology e of uniform convergence on the sets in \mathcal{E} is a locally convex topology on M_c . The following can then be demonstrated.

THEOREM 2.1. *Let X be a completely regular Hausdorff space, and let M_c be endowed with the topology e . Then the following hold:*

1. M_c is complete.
2. L is dense in M_c .
3. The dual space of M_c is C .
4. The restriction of the topology e to X is the original topology on X .

In order to prove the theorem, one first shows that the topology e on L is the finest locally convex topology on L whose restriction to X is the original topology and that the dual space of (L, e) is C . Again by combining Grothendieck's characterization of complete locally convex topologies with results from the theory of measures on topological spaces, one proves that the completion of (L, e) is (M_c, e) .

An application of Theorem 2.1 yields the following analogue of Granirer's theorem (Theorem 1.2). The same method of proof is used here.

THEOREM 2.2. *Let X be a completely regular Hausdorff space. Then the following hold:*

1. On M_c^+ the topologies e and $\sigma(M_c, C)$ are identical.
2. If B is a pointwise bounded, equicontinuous subset of C , then B is relatively $\sigma(C, M_c)$ -compact.

Another application of Theorem 2.1 yields a simple proof of the following theorem of Shirota. (See [1, p. 229].)

THEOREM 2.3. *Let X be completely regular Hausdorff. Then there is a complete uniform structure compatible with the topology of X if and only if X is realcompact.*

The same methods can be used to explore the closure of L in other natural topologies. For instance, the completion of L for the topology of uniform convergence on the norm compact subsets of C^b is the space M . Also if $M_\sigma(M_c)$ is given the Mackey topology for the pair (M_σ, C^b) $((M_c, C))$, then it is complete and L is a dense subspace. Furthermore, if X is realcompact, then the completion of L with the Mackey topology of the pair (L, C) is M_c .

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