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THE SPECTRUM OF AN AUTOMORPHISM

BY STEPHEN SCHEINBERG[1]

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In a series of articles H. Kamowitz and I investigated the nature of $\sigma(T)$, the spectrum of an arbitrary automorphism of an arbitrary semisimple commutative Banach algebra. This study was begun as a by-product of [**1**], in which we made the incidental observation that $\sigma(T)$ must meet $\{z:|z - 1| \geqq 1\}$, unless $T = I$. The following is a summary of the known necessary conditions (N) and the known sufficient conditions (S) on $\sigma(T)$.

N1. If $T^k = I$ (some $k \geqq 1$), then $\sigma(T)$ = a union of subgroups of the group of kth roots of 1, [**2**].

S1. Every possibility consistent with N1 can occur (direct sums of rotations).

N2. If $T^k \neq I$ (all $k \geqq 1$), then $\sigma(T) \supseteq$ the unit circle, [**2**].

S2. It is common that $\sigma(T)$ = the unit circle, but $\sigma(T)$ can be an annulus, [**2**].

N3. If $T^k \neq I$ (all $k \geqq 1$), then $\sigma(T)$ must be connected, [**3**].

S3. The set of $\sigma(T)$'s is closed under the mapping $1/z$, and if $U = \bigcup_\alpha \sigma(T_\alpha)$ is bounded away from 0 and ∞, then $\bar{U}$ is $\sigma(T)$ for some T. If R is a bounded region such that $\{1 < |z| < a\} \subseteq R \subseteq \{1 < |z|\}$ and $\{1 < |z|\} - R$ is a semigroup under multiplication, then $\bar{R}$ is $\sigma(T)$ for some T. The hypothesis that R be connected may be weakened somewhat, [**3**].

The purpose of this note is to extend the set of constructions of [**3**] to include cases where $\sigma(T)$ is not the closure of its interior. The following theorem illustrates the technique of attaching a line segment to a region.

THEOREM. *Let $\sigma = \{z:1 \leqq |z| \leqq 2\} \cup \{z:2 \leqq z \leqq 3\}$. Then there is a semisimple Banach algebra A and an automorphism T of A such that $\sigma(T) = \sigma$.*

PROOF. In the outline which follows I have omitted several routine calculations. Let A be the set of all functions which are bounded and analytic on $\{1 < |z| < 2\}$ and C^∞ on $\{1.5 \leqq z \leqq 3\}$ and satisfy $|f^{(n)}(z)| \leqq B \max(1, n!(\log n)^n)$ for some $B < \infty$, all $n \geqq 0$, and $1.5 \leqq z \leqq 3$. Define $p(f) = \sup\{|f(z)|:1 < |z| < 2\} + \inf B$. It is clear that p is a norm for A and that A is complete with respect to p.

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Define $f * g = \sum_{-\infty}^{\infty} a_n b_n z^n$, where $f = \sum a_n z^n$ and $g = \sum b_n z^n$. When f and g belong to A, $f * g$ is analytic on $1 < |z| < 4$ and

$$f * g(z) = \frac{1}{2\pi i} \int_{|w|=1} f(w) g\left(\frac{z}{w}\right) \frac{dw}{w} \quad \text{for } 1 < |z| < 2.$$

It follows that $f * g \in A$ and $p(f * g) \leqq \text{const}\, p(f) \cdot p(g)$. Then $\|f\| = \text{const}\, p(f)$ defines a Banach algebra norm on A.

The mapping $f \to a_n$ is a homomorphism of A onto C for each n. If $a_n = 0$ for all n, then $f \equiv 0$: this is obvious for $1 < |z| < 2$; for $1.5 \leqq z \leqq 3$ it is a consequence of Carleman's theorem on quasi-analytic classes [**4**, Chapter 1], since the nth root of $n!(\log n)^n$ is asymptotic to $(n/e) \log n$. Thus, A is semisimple.

Because of the rapid growth of $n!(\log n)^n$, every function which is analytic on a neighborhood of σ belongs to A. Furthermore, if g is such a function and f is arbitrary in A, then $gf \in A$ and $\|gf\| \leqq \text{const}\, \|f\|$.

Define $T: A \to A$ by $Tf(z) = zf(z)$. T is an automorphism of A and $\sigma(T) \supseteq \sigma$. If $\lambda \notin \sigma$, use $g = 1/(z - \lambda)$ in the preceding paragraph and we see that $\sigma(T) = \sigma$.

Remark. The construction given above can be extended. As an illustration let us attach a new line segment to the old one. For example, let $\sigma' = \sigma \cup \{z : z = 3 + iy, 0 \leqq y \leqq 1\}$. Define A' to be all functions which are bounded and analytic on $\{1 < |z| < 2\}$, C^∞ on each interval $\{1.5 \leqq z \leqq 3\}$ and $\{3 + iy : 0 \leqq y \leqq 1\}$ with $|f^{(n)}| \leqq B \max(1, n!(\log n)^n)$ on both intervals, and satisfying the Cauchy-Riemann condition $(\partial/\partial x)^n f = (i^{-1}\partial/\partial y)^n f$ at $z = 3$. The rest of the proof continues now with very slight changes. (Observe that the Cauchy-Riemann condition guarantees that any function analytic on a neighborhood of σ' will belong to A' and that any member of A' which is 0 on σ will be 0 on σ'.)

With the method of the theorem, disjoint domains can be connected by line segments, subject to the semigroup requirement of S3, and these constructions may be combined with those of [3] and iterated to produce quite complicated $\sigma(T)$.

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Department of Mathematics, Stanford University, Stanford, California 94305

Current address: Department of Mathematics, University of California, Irvine, California 92664