BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 3, May 1972

A REMARK ON STRONG PSEUDOCONVEXITY FOR ELLIPTIC OPERATORS

BY VICTOR GUILLEMIN[1]

Communicated by Shlomo Sternberg, October 19, 1971

The purpose of this note is to give a kind of intrinsic characterization of second order elliptic operators.

Let $p(x, D)$ be an mth order elliptic operator defined on an open subset U of $\boldsymbol{R}^n$. Let $p_m(x, \xi)$ be its leading symbol. Let φ be a smooth function on U with the property that grad $\varphi \neq 0$ when $\varphi = 0$. The hypersurface, $\varphi = 0$, is said to be *strongly pseudoconvex* at a point $x \in \varphi^{-1}(0)$ if

$$(1) \qquad \sum \frac{\partial^2 \varphi}{\partial x_j \partial x_k} p_m^{(j)}(x, \xi)\overline{p_m^{(k)}}(x, \xi) + \tau^{-1} \operatorname{Im} \sum p_{m,k}(x, \xi)\overline{p^{(k)}(x, \xi)} > 0,$$

for all $\xi = \eta + i\tau \operatorname{grad} \varphi$, where $\eta \in \boldsymbol{R}^n$ and $0 \neq \tau \in \boldsymbol{R}$, satisfying the equations:

$$(2) \qquad p_m(x, \xi) = 0 = \sum p_m^{(j)}(x, \xi)\, \partial\psi/\partial x_j.$$

(See Hörmander [**2**, Chapter 8].)

If p is a second order operator and its leading symbol is real, then, for $\eta, N \in \boldsymbol{R}^n$ not multiples of each other, the equations

$$(3) \qquad p_2(x, \eta + \tau N) = 0 = \sum p_2^{(j)}(x, \eta + \tau N) N_j$$

have no solutions, so condition (1) is satisfied trivially. This proves

PROPOSITION 1. *If $p(x, D)$ is second order and its leading symbol is real, then every hypersurface is strongly pseudoconvex.*

In this note we will prove a result in the other direction, namely,

PROPOSITION 2. *If $n \geqq 3$ and every hypersurface is strongly pseudoconvex then $p(x, D)$ is second order.*

REMARK 1. If there exist vectors η, N satisfying (3) it is easy to construct a φ, with grad $\varphi(x) = N$, violating (1). Therefore for every surface with normal, N, at x to be strongly pseudoconvex at x it is necessary and sufficient that there be no η, N satisfying (3). Hence Proposition 2 can be reformulated as a simple algebraic assertion, namely,

PROPOSITION 3. *Let $p(\zeta)$, $\zeta \in \boldsymbol{C}^n$, be a homogeneous polynomial of degree m. Assume $n \geqq 3$, and assume p satisfies the following conditions:*

AMS 1970 *subject classifications*. Primary 35-XX.

[1] Supported by NSF GP-22927.

(4)
(a) *p has no real zeros except* 0.
(b) *For any fixed pair of vectors* ξ *and* η *in* R^n, ξ *and* η *not multiples of each other, the equation* $p(\xi + \tau\eta) = 0, \tau \in C$, *has simple roots.*

Then $m = 2$.

REMARK 2. It is easy to see that condition (4) is equivalent to the following condition:

$$\sum \frac{\partial p(\xi)}{\partial \xi^i} \bar{\xi}^i \neq 0 \quad \text{when } p(\xi) = 0, \xi \in C^n \neq 0. \tag{5}$$

REMARK 3. It is enough to prove Proposition 3 when $n = 3$. In fact choose coordinates so that the coefficient of ξ_1^m in $p(\xi)$ is nonzero. If the theorem is true in dimension 3 then $p(\xi_1, \xi_2, \xi_3, 0, \ldots, 0)$ is of degree 2 so $m = 2$.

We will assume from now on that $n = 3$ and that $p(\xi)$ satisfies condition (5). Since $p(\xi)$ is homogeneous in ξ its zero variety defines an algebraic curve, γ, in $P^2(C)$. By (5), grad $p(\xi) \neq 0$ when $p = 0$ so this curve is nonsingular.

We identify $P^2(C)$ with the set of one-dimensional subspaces of C^3. Given $l \in P^2(C)$, let $L_l = \{\xi \in C^3, \xi \in l\}$ and let L be the vector bundle on $P^2(C)$ whose fiber at L is L_l. Let $L_\gamma = L \mid \gamma$. Given $l \in P^2(C)$, let ξ be a nonzero vector on l and let

$$H_l = \left\{(a_1, a_2, a_3) \in C^3, \sum \frac{\partial p}{\partial \xi^i}(\xi) a_i = 0\right\}.$$

Let H be the vector bundle on γ whose fiber at l is H_l. By Euler's identity $L_\gamma \subset H$. Let $J = H/L_\gamma$.

LEMMA 1. *The Chern class* $c_1(J) = 0$.

PROOF. Let E be the trivial bundle over γ with fiber C^3. Let $\bar{L}_\gamma$ be the line bundle over γ whose fiber at $l \in P^2(C)$ is $\{\bar{\xi}, \xi \in l\}$. It is easy to see that $\bar{L}$ is the dual bundle of L so $c_1(\bar{L}) = -c_1(L)$. The composite bundle map $L \to E \to E/H$ is bijective by condition (5) so $c_1(E/H) = c_1(\bar{L})$. Since E is trivial, $0 = c_1(E) = c_1(L) + c_1(J) + c_1(E/H) = c_1(L) + c_1(J) - c_1(L) = c_1(J)$. Q.E.D.

Let T be the holomorphic tangent bundle of γ. There is a canonical identification $T \cong \mathrm{Hom}(L_\gamma, J)$, so $c_1(T) = c_1(J) - c_1(L_\gamma) = -c_1(L_\gamma)$. By Riemann-Roch, $c_1(T) = 2 - 2g$, g being the genus of γ (see Gunning [**1**, p. 110]) so $c_1(L_\gamma) = 2g - 2$. On the other hand $c_1(L_\gamma) = -m$.

(PROOF. Let ω be a linear functional on C^3. Then ω defines a section of L^* whose zero set is the hyperplane $\omega = 0$. The Chern class of $L^* \mid \gamma$ is the number of points in which this hyperplane intersects γ, which is just the degree of p for hyperplanes in general position.)

To summarize we have proved the equality

$$2g - 2 = -m. \qquad (6)$$

Since m is positive this can be satisfied only for $g = 0$ and $m = 2$. Q.E.D.

Bibliography

1. R. C. Gunning, *Lectures on Riemann surfaces*, Princeton Math. Notes, Princeton Univ. Press, Princeton, N.J., 1966. MR **34** #7789.

2. L. Hörmander, *Linear partial differential operators*, Die Grundlehren der math. Wissenschaften, Band 116, Academic Press, New York; Springer-Verlag, Berlin, 1963. MR **28** #4221.

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139