BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 2, March 1972

# SURFACES IN CONSTANT CURVATURE MANIFOLDS WITH PARALLEL MEAN CURVATURE VECTOR FIELD

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Communicated by James Serrin, July 8, 1971

I. **Statement of results.** For an $(n)$-dimensional Riemannian manifold $M^n$, isometrically immersed in an $(n+k)$-dimensional Riemannian manifold $M_{(c)}^{(n+k)}$ of constant sectional curvature $c$, let $H$ denote the mean curvature vector field of $M^n$. $H$ is a section of the normal bundle $NM^n$ of the immersion. When $n = 2$, $k = 1$, and $c = 0$ (a surface immersed in $E^3$), the requirement $|H| =$ constant is classical constant mean curvature. If $k > 1$, however, the condition $|H| =$ constant is usually too weak to prove reasonable generalizations of the classical theorems for surfaces of constant mean curvature in $E^3$. We investigate a stronger requirement on $H$; namely, that $H$ be parallel with respect to the induced connection in the normal bundle (for definitions, see II). Then using an analytic construction first employed by H. Hopf [**2**], we obtain

THEOREM 1. *A compact surface $M^2$ of genus 0 immersed in $M^4(c)$, $c \geqq 0$, upon which $H$ is parallel in the normal bundle, is a sphere of radius $1/|H|$.*

This generalizes a theorem of Hopf, who proved that the only immersed surfaces in $E^3$ of genus 0 with constant mean curvature are spheres [**2**, Chapter 7, §4]. For complete surfaces in $E^4$, we prove

THEOREM 2. *A complete surface $M^2$, immersed in $E^4$, whose Gauss curvature does not change sign, and for which $H$ is parallel in the normal bundle $NM^2$, is a minimal surface ($H \equiv 0$), a sphere, a right circular cylinder, or a product of circles $S^1(r) \times S^1(\rho)$, where $|H| = \frac{1}{2}(1/r^2 + 1/\rho^2)^{1/2}$.*

This extends a theorem of Klotz and Osserman for complete surfaces of constant scalar mean curvature in $E^3$ [**5**]. It can also be generalized to immersions into $\overline{M}^4_{(c)}$, $c \geqq 0$. Theorem 2 is proved in two steps. First we prove

THEOREM 3. *A piece of immersed surface $M^2$ in $E^4$, satisfying the conditions of Theorem 2 with $H \neq 0$, is either a sphere or it is flat ($K = 0$).*

Then we establish the following characterization of flat surfaces in $E^4$ with parallel mean curvature vector fields:

*AMS* 1970 *subject classifications.* Primary 53C40, 53A05, 53C45, 53B20; Secondary 53A10, 30A52.

*Key words and phrases.* Constant curvature manifolds, parallel mean curvature vector, constant mean curvature, complete surfaces, Codazzi equations, conformal coordinates, pseudo-umbilical, parallel normal bundle.

THEOREM 4. *A piece of immersed surface $M^2$ in $E^4$ with parallel mean curvature vector $H \neq 0$ and $K \equiv 0$ is a piece of $S^1(r) \times S^1(\rho)$: the product of two circles of radius r and $\rho$ with the standard flat immersion.* ($\rho$ *may be infinite, in which case we have a right circular cylinder.*)

Theorem 2 generalizes to immersions in $S^4(c)$.

Surfaces in $E^n$ which lie minimally in hyperspheres of radius $r$ have the same mean curvature vectors as the hypersphere, and consequently have parallel mean curvature vector fields. Such surfaces are pseudo-umbilic ($\varphi_3 \equiv 0$ in the lemma in II). In this case, Itoh [**3**] has proven a special case of Theorem 2 for immersions in $E^4$ (see also Chen, [**1**]). For minimal surfaces in $S^4$, Ruh [**8**] has proven a case of Theorem 1, using methods similar to the basic lemma in II. For a wide variety of examples of complete minimal surfaces in $S^3$, see Lawson [**6**].

It is possible to show the existence of surfaces in $E^n$ and $S^n(c)$ with parallel $H$ and $\varphi_3 \neq 0$ (i.e. they do not lie minimally in hyperspheres). The method employed uses a theorem due to Szczarba [**9**] on existence of immersions in constant curvature manifolds with codimension $k > 1$.

II. **Definitions and Main Lemma.** $\bar{\nabla}$ denotes covariant differentiation on $\bar{M}^{n+k}_{(c)}$, and $\nabla$ denotes covariant differentiation on $M^n \subset \bar{M}^{n+k}$. For $X$, $Y$, sections of $TM^n$, $\nabla_X Y = [\bar{\nabla}_X Y]^T$ where $[\ ]^T$ is projection onto $TM^n$. $[\ ]^N$ is projection onto $NM^n$.

DEFINITIONS. $B(X, Y) = [\bar{\nabla}_X Y]^N$. $B$ is the second fundamental form tensor of the immersion. Similarly for $N$, a section of $NM^n$, $D_X N = [\bar{\nabla}_X N]^N$. $D$ defines a connection on $NM^n$. $A(X, N) = [\bar{\nabla}_X N]^T$. $A$ is a tensor: $A_p$: $TM^n \times NM^n \to TM^n$ is bilinear.

For an orthonormal framing $(e_1 \cdots e_n)$ of $TM^n$, $H = (1/n)\sum_{i=1}^n B(e_i, e_i)$. This definition of $H$ is independent of the framing. A normal vector field $N$ is said to be parallel in the normal bundle $NM^n$ if $D_X N = 0$ for all $X$ in $TM^n$. This condition implies $|N| =$ const. Thus our assumption that $H$ is parallel in $NM^n$ includes constant mean curvature. ($|H| = c$.)

The Gauss and Codazzi equations, for $X$, $Y$, $Z$ sections in $TM^n$, are

$$(1)\quad R(X,Y)Z = c\{\langle X,Z\rangle Y - \langle Y,Z\rangle X\} + A(X,B(Y,Z)) - A(Y,B(X,Z)),$$

$$(2)\qquad (\nabla_X B)(Y,Z) = (\nabla_Y B)(X,Z),$$

where $(\nabla_X B)(Y,Z) = D_X(B(Y,Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z)$ (for a reference for the above definitions and equations, see [**4**, Chapter 7]).

For $X$, $Y$ in $TM^n$, $N$ in $NM^n$, $\tilde{R}(X, Y)N = D_X D_Y N - D_Y D_X N - D_{[XY]}N$ is the curvature tensor for $D$. For $\tilde{R}$, there is a Gauss-type equation

$$(3)\qquad \tilde{R}(X,Y)N = B(X, A(N,Y)) - B(Y, A(N,X))$$

and an equation, analogous to (2),

$$(4)\qquad (\nabla_X A)(Y,N) = (\nabla_Y A)(X,N).$$

For a unit normal vector $e_\alpha$ at $p \in M^n$, the matrix $(\lambda^\alpha_{ij}) = (B(e_i, e_j) \cdot e_\alpha)$ is the "second fundamental form matrix in the $\alpha$ direction." We specify $H/|H|$ as $e_{n+1}$ when $H \neq 0$. Considering an immersed surface ($n = 2$) given in conformal coordinates $(u, v)$: $ds^2 = E(du^2 + dv^2)$, we associate to it a natural framing

$$(e_1, e_2) = \left(\frac{\partial}{\partial u}\Big/\sqrt{E}, \frac{\partial}{\partial v}\Big/\sqrt{E}\right)$$

of the tangent bundle, $TM^2$.

LEMMA. *For an immersed surface, $M^2 \hookrightarrow \overline{M}^{n+k}_{(c)}$ in conformal coordinates, let $H \neq 0$ and $e_\alpha$ be a unit normal vector field with $e_\alpha \cdot H = 0$:*

(a) *if $H$ is parallel in $NM^2$, then $\varphi_3 = E\{\frac{1}{2}(\lambda^3_{11} - \lambda^3_{22}) - i\lambda^3_{12}\}$ is an analytic function of $z = u + iv$;*

(b) *if $e_\alpha$ is parallel in $NM^2$, then $\varphi_\alpha = E\{\lambda^\alpha_{11} - i\lambda^\alpha_{12}\}$ is an analytic function of $z$;*

(c) *if $k = 2$ and $H$ is parallel, then $e_\alpha$ is parallel, and both $\varphi_3$ and $\varphi_\alpha$ are analytic;*

(d) *under the conditions of* (a) *and* (b), *either $\varphi_3 \equiv 0$ or $\varphi_\alpha = \kappa\varphi_3$ where $\kappa$ is a real constant.*

SKETCH OF PROOF. (a) Using equation (4) with $X = \partial/\partial u$, $Y = \partial/\partial v$, $N = H$, and the assumption that $H$ is parallel, the equations

$$(5)\quad (E\lambda^3_{11})_v - (E\lambda^3_{12})_u = \tfrac{1}{2}E_v(\lambda^3_{11} + \lambda^3_{22}), \qquad (E\lambda^3_{12})_u - (E\lambda^3_{22})_v = \tfrac{1}{2}E_u(\lambda^3_{11} + \lambda^3_{22})$$

are obtained. (5) is in the same form as the Codazzi equations in conformal coordinates for surfaces in $E^3$, only it is expressed for the distinguished normal $e_3 = H/|H|$. Since $\lambda^3_{11} + \lambda^3_{22} = 2|H| =$ constant, (5) reduces to the Cauchy-Riemann equations for $\varphi_3$.

(b) Proof follows that of (a), using the fact that $\lambda^\alpha_{11} + \lambda^\alpha_{22} = 0$.

(c) Since $NM^2$ is 2-dimensional, the assumption that $H$ is parallel forces $e_\alpha$ to be parallel. Then (a) and (b) imply analyticity.

(d) Using equation (3) with

$$X = \frac{\partial}{\partial u_1}\Big/\sqrt{E}, \qquad Y = \frac{\partial}{\partial u_2}\Big/\sqrt{E}, \quad \text{and} \quad N = e_3,$$

we obtain, using the fact that $e_3$ is parallel,

$$(6) \qquad 0 = \left(\sum_{k=1}^{2} \lambda^3_{k2}\lambda^\alpha_{k1} - \lambda^3_{k1}\lambda^\alpha_{k2}\right).$$

Note that (6) implies $\mathrm{Im}(\varphi_\alpha \cdot \bar{\varphi}_3) = 0$. So if $\varphi_3 \not\equiv 0$, $\varphi_\alpha/\varphi_3 = \varphi_\alpha \cdot \bar{\varphi}_3/|\varphi_3|^2$ is real. By (a) and (b), it is also meromorphic, hence constant.

III. **Proof of Theorems (Sketch).** Theorem 1 is proved by constructing an analytic differential $\theta_3$ out of the functions $\varphi_3(z)$ of the lemma: in local coordinates, $\theta_3 = \varphi_3 dz^2$. Since $M^2$ is of genus 0, $\theta_3$ must be identically zero.

Hence $\varphi_3(z) \equiv 0$. This implies that $M^2$ is pseudo-umbilic ($\lambda^3_{11} = \lambda^3_{22}$, $\lambda^3_{12} = 0$). The function $\varphi_4$ associated with $e_4, e_4 \cdot H = 0$ is also zero by a similar argument. Hence $M^2$ is totally umbilic. This implies that $M^2$ is immersed as a sphere.

To prove Theorem 3, we can consider on $M^2$ the quadratic analytic differentials $\theta_3$ and $\theta_4$ given locally by $\varphi_3 dz^2$ and $\varphi_4 dz^2$ (where $\varphi_3$, $\varphi_4$, and $z$ are as in the lemma). If $K \geqq 0$, $M^2$ is either compact or parabolic by a theorem of Huber (see [**5**, p. 316]). If it is compact, then either $K \equiv 0$ or $M^2$ is of genus 0. The genus 0 case is a sphere by Theorem 1.

If $K \leqq 0$, then $|H|^2 - K > |H|^2 > 0$. In a neighborhood of each point, we introduce the new metric $d\tilde{s}^2 = E(|H|^2 - K)^{1/2}(du^2 + dv^2)$. Using the equality

$$|\varphi_3|^2 + |\varphi_4|^2 = E^2(|H|^2 - K)$$

and part (d) of the lemma to show that $\Delta \log(|\varphi_3|^2 + |\varphi_4|^2) = 0$, we establish that $d\tilde{s}^2$ is a flat metric. Therefore, the univeral covering surface $\tilde{M}^2$ of $M^2$ is conformally the plane. The function $|H|^2 - K$ is easily seen to be superharmonic. Since it is bounded below, it must be constant. Hence $K$ is constant, and must be zero.

Theorem 4 is proved by introducing conformal coordinates $(u, v)$ such that $E \equiv 1$. The lemma is used to show that all $\lambda^\alpha_{ij}$ are constant. Then a rotation of coordinates puts the immersion into the form

$$(u, v) \to \left(r \cos \frac{u}{r}, r \sin \frac{u}{r}, \rho \cos \frac{v}{\rho}, \rho \sin \frac{v}{\rho}\right).$$

The constants $r$ and $\rho$ are determined from the $\lambda^\alpha_{ij}$ and $|H|$. This immersion is the standard flat immersion of the plane into $E^4$ as a product of circles.

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